Topological asymptotic expansions for a class of quasilinear elliptic equations. Estimates and asymptotic expansions of condenser p-capacities. The anisotropic case of segments.

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THÈSE

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Résumé

Mots-clés : *équations elliptiques quasilinéaires, analyse asymptotique topologique, coexistence de deux normes, problème d'interface non linéaire, p-capacités, obstacles de codimension* ≥ 2.

Les développements asymptotiques topologiques n'ont pas encore été étudiés pour les équations elliptiques quasilinéaires, comme celle de p-Laplace. Cette question apparaît dans la perspective d'appliquer les méthodes d'asymptotique topologique en optimisation de forme aux équations non linéaires de l'élasticité comme en imagerie pour la détection d'ensembles de codimension ≥ 2 (points en 2D ou segments en 3D).

Dans la Partie I, notre principal résultat réside dans l'obtention du développement asymptotique topologique pour une classe d'équations elliptiques quasilinéaires, perturbées dans des sous-domaines non vides. Le gradient topologique peut être décomposé en un terme linéaire classique et en un terme nouveau, qui rend compte de la non linéarité.

L'étude des difficultés spécifiques qui apparaissent avec l'équation de p-Laplace, par comparaison avec l'équation de Laplace, montre qu'un point central réside dans la possibilité de définir la variation de l'état direct à l'échelle 1 dans $\mathbb{R}^N$. Nous étudions en conséquence des espaces de Sobolev à poids et quotientés, dont la semi-norme est la somme des normes L^p et L^2 du gradient dans $\mathbb{R}^N$. Puis nous construisons une classe d'équations elliptiques quasilinéaires, telle que le problème définissant l'état direct à l'échelle 1 vérifie une double propriété de p et 2 ellipticité. La méthode se poursuit par l'étude du comportement asymptotique de la solution du problème d'interface non linéaire dans $\mathbb{R}^N$ et par une mise en dualité appropriée des états directs et adjoints aux différentes étapes d'approximation. Avant d'obtenir enfin le développement asymptotique topologique, il faut préalablement estimer asymptotiquement des quantités de la forme $\| \,.\, \|_p^p + \| \,.\, \|_2^2$ pour les variations de l'état direct.

La Partie II traite d'estimations et de développements asymptotiques de p-capacités de condensateurs, dont l'obstacle est d'intérieur vide et de codimension ≥ 2.

Après quelques résultats préliminaires, nous introduisons les *condensateurs équidistants* pour étudier le cas des segments. L'effet anisotrope engendré par un segment dans l'équation de p-Laplace est tel que l'inégalité de réarrangement de Pólya-Szegö pour les intégrales de type Dirichlet fournit un minorant trivial. De plus, quand $p > N$, on ne peut construire par extension une solution admissible pour le segment, aussi petite sa longueur soit-elle, à partir du cas du point.

Nous établissons une minoration de la p-capacité N-dimensionnelle d'un segment, qui fait intervenir les p-capacités d'un point, respectivement en dimensions N et $(N-1)$. Les cas de positivité de la p-capacité s'en déduisent. Notre méthode peut être étendue à des obstacles de dimensions supérieures et de codimension ≥ 2.

Introduisant les *condensateurs elliptiques*, nous montrons que le gradient topologique de la 2-capacité n'est pas un outil approprié pour distinguer les courbes et les obstacles d'intérieur non vide en 2D. Une solution pourrait être de choisir différentes valeurs de p ou bien de considérer le développement asymptotique à l'ordre 2, i.e. la *hessienne topologique*.

Le lecteur trouvera en Partie III la synthèse du manuscrit en langue française.

Abstract

Keywords: *quasilinear elliptic equations, topological asymptotic analysis, two-norms discrepancy, nonlinear interface problem, p-capacities, obstacles with codimension* ≥ 2.

Topological asymptotic expansions for quasilinear elliptic equations, such as the p-Laplace equation, have not been studied yet. Such questions arise from the need to apply topological asymptotic methods in shape optimization to nonlinear elasticity equations as in imaging to detect sets with codimensions ≥ 2 (e.g. points in 2D or segments in 3D).

In Part I our main contribution is to provide topological asymptotic expansions for a class of quasilinear elliptic equations, perturbed in non-empty subdomains. The topological gradient can be split into a classical linear term and a new term which accounts for the nonlinearity of the equation.

Comparing with steps carried out to obtain such expansions for the Laplace equation, it turns out that for the p-Laplace equation, one key point lies in the ability to define the variation of the direct state at scale 1 in $\mathbb{R}^N$. Accordingly we build dedicated weighted quotient Sobolev spaces, which semi-norms encompass both the L^p norm and the L^2 norm of the gradient in $\mathbb{R}^N$. Then we consider an appropriate class of quasilinear elliptic equations, to ensure that the problem defining the direct state at scale 1 enjoys a combined p and 2 ellipticity property. The asymptotic behavior of the solution of the nonlinear interface problem in $\mathbb{R}^N$ is then proven. An appropriate duality scheme is set up between direct and adjoint states at various stages of approximation. So as to prove eventually the desired topological asymptotic expansion, we first obtain asymptotic estimates of terms of the type $\| \, . \, \|_p^p + \| \, . \, \|_2^2$ for the variations of the direct state.

Part II deals with estimates and asymptotic expansions of condenser p-capacities and focuses on obstacles with empty interiors and with codimensions ≥ 2.

After preliminary results, we introduce *equidistant condensers* to study the case of segments. The anisotropy caused by a segment in the p-Laplace equation is such that the Pólya-Szegö rearrangement inequality for Dirichlet type integrals yields a trivial lower bound. Moreover, when $p > N$, one cannot build an admissible solution for the segment, however small its length may be, by extending the case of a punctual obstacle.

We provide a lower bound to the N-dimensional condenser p-capacity of a segment, by means of the N-dimensional and of the $(N-1)$-dimensional condenser p-capacities of a point. The positivity cases follow for p-capacities of segments. Our method can be extended to obstacles of higher dimensions and with codimension ≥ 2.

Introducing *elliptical condensers*, we show that the topological gradient of the 2-capacity is not an appropriate tool to separate curves and obstacles with non-empty interior in 2D. One way out could be to consider different values of parameter p or to consider the second order of the topological expansion, i.e. the *topological hessian*.

Contents

Chapter 1

Introduction and overview

The present manuscript is devoted to the study of

1. topological asymptotic expansions for quasilinear elliptic equations of second order, perturbed in non-empty subdomains;
2. estimates and asymptotic expansions of p-capacities of condensers, in particular for obstacles with empty interior, such as points and segments.

1.1 Motivations and goals

The methods of so-called *topological asymptotic expansions* or *topological gradients* or *topological sensitivity* have been developed since the 1990's [78, 63, 44, 68, 81]. They are applied in the field of shape optimization (e.g. [81, 3, 14, 83, 58]) as well as in image processing (e.g. [20, 21, 22, 23, 24, 25, 26, 57, 15, 58]).

The key idea is to assess the sensitivity of an appropriately chosen functional taken on the solution of a partial differential equation, when the latter is perturbed in the vicinity of a given point x_0, in a subdomain of which one geometric parameter goes down to zero. More precisely, let $\Omega \subset \mathbb{R}^N$ a bounded domain. Let a partial differential equation in Ω, e.g. the Laplace equation, with a boundary condition on boundary $\partial\Omega$, e.g. a Dirichlet or a Neumann condition, and assume it admits a unique solution u_0, called the unperturbed *direct state*, in an appropriate functional space $\mathcal{F}_0$. Let $\omega \subset \mathbb{R}^N$ a bounded domain containing the origin 0. Let a point $x_0 \in \Omega$ and $\varepsilon > 0$ small enough such that $\omega_\varepsilon := x_0 + \varepsilon\, \omega \subset \Omega$. Then, as shown on Figure 1.1, modify the equation in ω_ε, either by changing a coefficient of the equation in ω_ε, for instance a conductivity, or by restricting the domain of the equation to $\Omega_\varepsilon := \Omega \setminus \bar{\omega}_\varepsilon$ and by requiring a boundary condition on boundary $\partial\omega_\varepsilon$. Assume that the perturbed equation obtained that way admits a unique solution u_ε, called the perturbed *direct state*, in a functional space $\mathcal{F}_\varepsilon$.

Let $J_\varepsilon : \mathcal{F}_\varepsilon \to \mathbb{R}$ a functional defined for $\varepsilon \geq 0$ small enough. Then the *topological asymptotic expansion*, when one holds, is of the form

$$J_\varepsilon(u_\varepsilon) = J_0(u_0) + \rho(\varepsilon)\ g(x_0) + o(\rho(\varepsilon)), \quad \forall\, \varepsilon \geq 0 \text{ small enough}, \tag{1.1.1}$$

where ρ is a non negative function such that $\lim\limits_{\varepsilon\to 0} \rho(\varepsilon) = 0$. The scalar $g(x_0)$ is called the *topological gradient* at point x_0. Hence one defines the *topological gradient* $g : \Omega \to \mathbb{R}$.

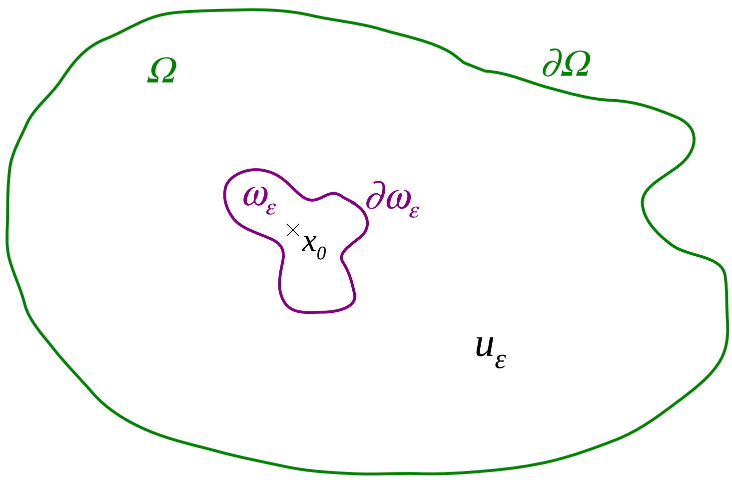

Figure 1.1: An equation perturbed in ω_ε

The signs and absolute values taken by function g in Ω are the decisional input used by a dedicated algorithm, depending on the applicative task to perform.

When the perturbation ω_ε is anisotropic, e.g. if ω_ε is a segment $\subset \mathbb{R}^2$, which external normal is denoted by n, then it holds [9]

$$g(x_0) = -\nabla u_0(x_0) A(n) \nabla v_0(x_0),$$

where v_0 denotes the so-called unperturbed *adjoint state*. In such a case, decisional information is provided by eigenvectors and eigenvalues of *polarization matrix* $A(n)$ and by the two vector fields ∇u_0 and ∇v_0.

Topological asymptotic expansions were obtained for many equations such as linear elasticity equations [44], Helmholtz equation [77], Stokes equations [48] and Navier-Stokes equations [10].

Regarding the Laplace equation, topological asymptotic expansions were obtained in the case of Dirichlet boundary condition [47, 11] as in the case of Neumann boundary condition [70, 13]. Moreover asymptotic expansions were provided for the first eigenvalues and eigenfunctions of classical problems for the Laplace operator perturbed in small domain ω_ε in 2 and 3 dimensional domains [66, 68].

Hence from a mathematical perspective, the question of topological asymptotic expansions for nonlinear elliptic equations of second order is one which naturally arises next. The case of semilinear equations, made of the Laplace operator added to a nonlinear term, was studied in [11, 50]. But to our best knowledge, topological asymptotic expansions remain unknown for nonlinear elliptic equations, with nonlinear differential operators, such as quasilinear equations and in particular the p-Laplace equation ($p \neq 2$).

Moreover such questions also arise from at least two applicative fields.

1. In the field of shape optimization, the use of linear elasticity equations remains a drawback whenever the actual behavior of mechanical structures is better des-

cribed by equations of nonlinear elasticity [35]. This issue was raised e.g. in [2] §8.

A simplified model of nonlinear elasticity is provided by [38] §5.7.2 as follows. Let $p > 1$ and q the Hölder conjugate exponent of p. Let $F \in L^q(\Omega, \mathbb{R}^N)$. The solution u is defined by

$$\{u\} := \operatorname{argmin} \left\{ J(v); v \in W_0^{1,p}(\Omega, \mathbb{R}^N) \right\} \tag{1.1.2}$$

where $\mathcal{J}$ is given by

$$\mathcal{J}(v) := \int_\Omega \left[\frac{1}{p} |\epsilon(v)|^p - F.v \right]$$

and

$$\epsilon_{i,j}(v) = \frac{1}{2} (\partial_j v_i + \partial_i v_j) \quad \text{and} \quad |\epsilon(v)|^p = \left(\sum_{i \le j} |\epsilon_{i,j}(v)|^2 \right)^{\frac{p}{2}}.$$

Though this problem is 3-dimensional with obvious interactions between the three dimensions due to the definition of $\epsilon_{i,j}(v)$, it has the minimization of a Dirichlet type integral in common with the scalar p-Laplace problem with Dirichlet boundary conditions. The existence and uniqueness of the solution u follows from the strict convexity of the functional and from the coercivity provided by the Korn inequality ([38], chap. 7).

Therefore as a first step it makes sense to study topological asymptotic expansions for the p-Laplace equation or more generally for quasilinear elliptic equations.

2. In the field of imaging, the detection of subsets of codimension ≥ 2, as points in $2D$ as curves in $3D$, remains an important task, e.g. in medical imaging. A smooth curve can be locally approximated by a segment, of length 'small enough'. Applying a topological asymptotic method, the task of detecting segments in $2D$ images has been dealt with in [9] using a Laplace equation with Neumann boundary condition. According to the theory of potential [1, 49], the Laplace equation can only detect subsets which codimensions are < 2. For instance it cannot detect points in $2D$ or segments in $3D$. For such tasks, one may consider the p-Laplace equation, where parameter p is chosen strictly larger that the codimension of the subsets to detect.

Depending on such motivations, the present PhD work first focused on the study of estimates and asymptotic expansions of p-capacities of condensers. In other words, we studied the p-Laplace equation with Dirichlet homogeneous boundary condition, when it is perturbed by an obstacle in which the solution is required to be ≥ 1. We emphasized cases in which the obstacle has an empty interior, such as a point or more importantly a segment. Due to a descending continuity property it matters to understand how such cases may be approximated by condensers of which the obstacle has a non-empty interior, such as balls to approximate points or ellipsoids to approximate segments. Few standard methods may be applied there.

At a second stage, we initiated the study of topological asymptotic expansions for quasilinear linear equations of second order with a source applied in the domain and with a Dirichlet homogeneous boundary condition. The perturbation is caused by a variation of the conductivity in subset ω_ε. Although the research process has been anything but straight, it soon turned out that the proper implementation of an asymptotic analysis, and in particular the definition of the variation of the direct state at scale 1 in $\mathbb{R}^N$, would first require to build an appropriate functional framework and to choose accordingly an appropriate class of quasilinear equations.

1.2 Structure of the manuscript

This manuscript is divided into two rather independent parts as follows.

1. Part I starting on page 23 deals with topological asymptotic expansions for quasilinear elliptic equations of second order, perturbed in subsets with non-empty interior.
2. The study of estimates and asymptotic expansions of p-capacities of condensers, especially for points and segments, will be found in Part II starting on page 107.

In part I, we first analyze in chapter 2 some of the specific issues arising in the process of obtaining topological asymptotic expansions for quasilinear elliptic equations. To serve as a reference and considering the Laplace equation, we sketch the steps usually taken for that purpose in the case of a linear elliptic equation. Then in the case of the p-Laplace equation, we emphasize the conditions required to define the variation of the direct state at scale 1 in $\mathbb{R}^N$. These conditions justify that we build dedicated weighted quotient Sobolev spaces in chapter 3. They also partly determine the class of quasilinear elliptic equations which is introduced in chapter 4 to obtain topological asymptotic expansions.

Hence chapter 3 is devoted to the construction of reflexive Banach spaces denoted $\mathcal{W}(\mathbb{R}^N)$ and $\mathcal{V}(\mathbb{R}^N)$ and of a Hilbert space $\mathcal{H}(\mathbb{R}^N)$, all of them enjoying Poincaré inequalities. The building scheme is rather standard. The main difficulty lies in the fact that in the nonlinear case the quotient space cannot be identified with a closed subspace of the weighted Sobolev space. This issue is overcome by Proposition 3.3.2 which allows to complete the proof of the Poincaré inequality. Eventually Proposition 3.5.2 will be pivotal to ensure a combined p- and 2-coercivity to the nonlinear operator defining the variation of the direct state at scale 1.

In chapter 4, we first define in section 4.1 a class of non-quadratic potentials which satisfy a combined p- and 2-ellipticity property as expected after the conclusions of chapter 2. Then section 4.2 describes the perturbed quasilinear elliptic equation we consider.

Our main contribution is the topological asymptotic expansion stated in Theorem 4.3.1, section 4.3 on page 56. To obtain such result, we study the direct state and its variations at different stages of approximation in section 4.4. The study of the adjoint state is available in section 4.5. The steps taken for the adjoint state are classical as we define the adjoint state as solution of a linearized equation. By contrast the nonlinear approach applied to the direct state is fairly new.

Then we prove the topological asymptotic expansion of the functional in section 4.6, separating a linear term and a nonlinear term. While both terms depend on the variations of the direct and adjoint states at scale 1 in $\mathbb{R}^N$, one essential ingredient for the nonlinear term is an operator denoted S characterizing the nonlinearity of the considered equation.

Part II on page 107 deals with estimates and asymptotic expansions of p-capacities of condensers. We first recall the context, some definitions and our goals in section 5.1.

In section 5.2, we provide some preliminary tools for estimation of p-capacities of condensers when the obstacle has a non-empty interior.

We apply the latter results in section 5.3 to obtain directly the positivity rule for condenser p-capacities of a point and to estimate the speed of convergence of the descending continuity property.

We then turn to condenser p-capacities of segments in section 5.4. For this purpose, we first introduce so-called *equidistant condensers*. We give two illustrations of the strong anisotropy of the problem, first applying the Pólya-Szegö rearrangement inequality for Dirichlet type integrals and then trying to derive an admissible solution for a segment of length $\varepsilon > 0$ 'small enough' from the solution of a spherical condenser.

We then provide a lower bound to the condenser p-capacity of a segment in a N-dimensional bounded domain, by means of the N-dimensional p-capacity of a point and of the $(N-1)$-dimensional p-capacity of a point. We can then prove directly the positivity rule for the condenser p-capacity of a segment in a N-dimensional bounded domain.

For the purpose of further estimation, we introduce *elliptical condensers*, defined in elliptic coordinates. The angular coordinate ν so to speak makes the dimension in which operates the p-Laplace equation, continuously change from N for $\nu = 0$ to $(N-1)$ for $\nu = \pi/2$ and then back to N for $\nu = \pi$. As variables are separable when $p = 2$, we obtain an estimate and the asymptotic expansion for the 2-capacity of a segment in a 2-dimensional bounded domain. It turns out that the first term of the expansion of such a 2-capacity is unable to separate curves and obstacles with non-empty interior in the plane.

Part I

Topological asymptotic expansions for quasilinear elliptic equations

Notation for Part I

In all Part I, encompassing chapters 2, 3 and 4, let $N \in \mathbb{N}$, $N \geq 2$.
Let $p \in [2, \infty)$, and q the Hölder conjugate exponent of p defined by $1/p + 1/q = 1$.
Classical notation will be used as follows:

1. The symbol $|E|$ denotes either the usual euclidean norm of E in $\mathbb{R}^N$ when $E \in \mathbb{R}^N$, or the N-dimensional Lebesgue measure of E when $E \subset \mathbb{R}^N$.
2. For all $a > 0$, we denote $B_a := \left\{x \in \mathbb{R}^N ; |x| < a\right\}$ and $B'_a := \mathbb{R}^N \setminus \overline{B}_a$.
3. S^{N-1} will be the unit sphere in $\mathbb{R}^N$ and A^{N-1} its surface area.
4. I_N denotes the N-dimensional identity matrix.
5. For all open subset $\mathcal{O} \subset \mathbb{R}^N$ or $\mathcal{O} \subset \mathbb{R}$, $C_0^\infty(\mathcal{O})$ denotes the space of infinitely differentiable functions with compact support $\subset \mathcal{O}$ and $\mathcal{D}'(\mathcal{O})$ denotes the space of distributions in $\mathcal{O}$.
6. The topological dual of a normed space $\mathcal{F}$ is denoted $\mathcal{F}^*$.

Moreover let Ω a bounded domain of $\mathbb{R}^N$. We denote

1. $W^{1,p}(\Omega)$ the Sobolev space defined by
$$W^{1,p}(\Omega) := \{u \in \mathcal{D}'(\Omega); u \in L^p(\Omega), \nabla u \in L^p(\Omega)\}$$
endowed with the norm
$$\|u\|_{1,p} := \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} ;$$
2. $\mathcal{V} := W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$;
3. $H^1(\Omega)$ the Hilbert space defined by
$$H^1(\Omega) := \left\{u \in \mathcal{D}'(\Omega); u \in L^2(\Omega), \nabla u \in L^2(\Omega)\right\}$$
endowed with the norm
$$\|u\|_{1,2} := \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} ;$$
4. $\mathcal{H} := H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H_1(\Omega)$.

Chapter 2

Specific issues arising for a quasilinear elliptic equation

In this chapter, we analyze specific issues which arise in the process of obtaining a topological asymptotic expansion for a second order quasilinear elliptic equation by comparison with a linear elliptic equation. To serve as a reference, we first briefly recall the main steps taken to obtain the topological asymptotic expansion in the case of a linear elliptic equation. Then we focus on conditions allowing to apply the Minty-Browder Theorem ([61], Chap. 2 §2, [30] Thm. V.15), so as to give sense to the variation of the direct state at scale 1. In the last section 2.4, we summarize specific issues encountered with a quasilinear elliptic equation.

To our best knowledge, putting aside quasilinear elliptic equations which actually are linear equations such as the Laplace or the Helmholtz equations, no topological asymptotic expansion was previously obtained for a (genuinely nonlinear) quasilinear elliptic equation. Only two articles deal with topological asymptotic expansions for semilinear equations, as follows:

- In [11], Amstutz studied the topological sensitivity for a class of nonlinear equations of the form
$$-\tilde{\Delta} u + \Phi(u) = \sigma, \quad u \in H_0^1(\Omega)$$
where $-\tilde{\Delta}$ is a linear and homogeneous differential operator of order 2 and Φ is a possibly nonlinear function mapping. The functional setting is one of Hilbertian spaces.
- In [50], Iguernane *et al.* studied topological derivatives for semilinear elliptic equations of the form
$$\begin{cases} -\Delta u^\varepsilon = F(x, u^\varepsilon(x)), & \text{in } \Omega(\varepsilon) \\ u^\varepsilon = 0, & \text{on } \Omega(\varepsilon) \end{cases}$$
where $\omega_\varepsilon := \varepsilon\,\omega$, $\Omega(\varepsilon) := \Omega\backslash\overline{\omega}_\varepsilon$, Δ is the Laplacian operator and $F \in C^{0,\alpha}(\Omega\times\mathbb{R})$. The functional setting is one of weighted Hölder spaces.

In the subsequent, *second order quasilinear equations* $Qu = 0$ will be defined according to [46], Chap. 10, i.e. operator Q is of the form
$$Qu = \sum_{i,j=1}^{N} a^{i,j}(x, u, \nabla u)\partial^2_{i,j} u \;+\; b(x, u, \nabla u), \qquad \text{with } a^{i,j} = a^{j,i},$$

where Ω is a domain of $\mathbb{R}^N$, $N \geq 2$, $x \in \Omega$, and $a^{i,j}(x,z,\xi)$, $i,j = 1, ..., N$, and $b(x,z,\xi)$ are defined for all $(x,z,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

Operator Q is said to be *elliptic* in $\Omega \times \mathbb{R} \times \mathbb{R}^N$ if matrix $[a^{i,j}(x,z,\xi)]$ is positive for all $(x,z,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

In Part I, we shall study quasilinear equations which are Euler-Lagrange equations of functionals of the form

$$\int_\Omega [\gamma W(\nabla u) - fu], \tag{2.0.1}$$

where $\gamma : \Omega \to \mathbb{R}_+^*$ is a *positive conductivity function*, $W \in C^1(\mathbb{R}^N, \mathbb{R})$ is called the *potential* and $f : \Omega \to \mathbb{R}$ is a *source*.

Denote the gradient field $T := \nabla W \in C^0(\mathbb{R}^N, \mathbb{R}^N)$. Under relevant assumptions, in an appropriate functional space and with appropriate boundary condition, a function u minimizes functional (2.0.1) if and only if it satisfies the following Euler-Lagrange equation

$$Qu = -\operatorname{div}(\gamma\, T(\nabla u)) - f = 0, \tag{2.0.2}$$

(see [41], Chap. 8).

Operator Q is *nonlinear* if T is not an affine vector field or equivalently if W is not a quadratic function.

For a given $p \in (2, \infty)$, we shall in particular study the two following cases of elliptic quasilinear equations:

1. When the potential is defined by $W(\varphi) := \frac{1}{p} |\varphi|^p$, $\forall \varphi \in \mathbb{R}^N$. It leads to considering the functional

   $$\int_\Omega \left[\frac{\gamma}{p} |\nabla u|^p - fu\right]$$

   and the following form of p-Laplace equation

   $$-\operatorname{div}\left(\gamma |\nabla u|^{p-2} \nabla u\right) - f = 0. \tag{2.0.3}$$

2. When, for a given real number $a > 0$, the potential is defined by

   $$W_a(\varphi) := \frac{1}{p}\left(a^2 + |\varphi|^2\right)^{\frac{p}{2}}, \quad \forall \varphi \in \mathbb{R}^N.$$

   It leads to considering the functional

   $$\int_\Omega \left[\frac{\gamma}{p}\left(a^2 + |\nabla u|^2\right)^{\frac{p}{2}} - fu\right]$$

   and the following quasilinear equation

   $$-\operatorname{div}\left(\gamma \left(a^2 + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u\right) - f = 0. \tag{2.0.4}$$

From the perspective of topological asymptotic expansions, we shall see that properties of equations (2.0.3) and (2.0.4) broadly differ, far beyond the well-known fact that equation (2.0.3) is degenerate while equation (2.0.4) is not.

2.1 Standard steps taken for a linear elliptic equation

In order to set up the reference situation, let's first recall formally and in a simplified way the main steps taken for obtaining the topological asymptotic expansion of a linear elliptic equation ([8, 63, 68]). This section 2.1 is in particular inspired by [12]. We consider the perturbation of the conductivity in the Laplace equation with Dirichlet homogeneous boundary conditions.

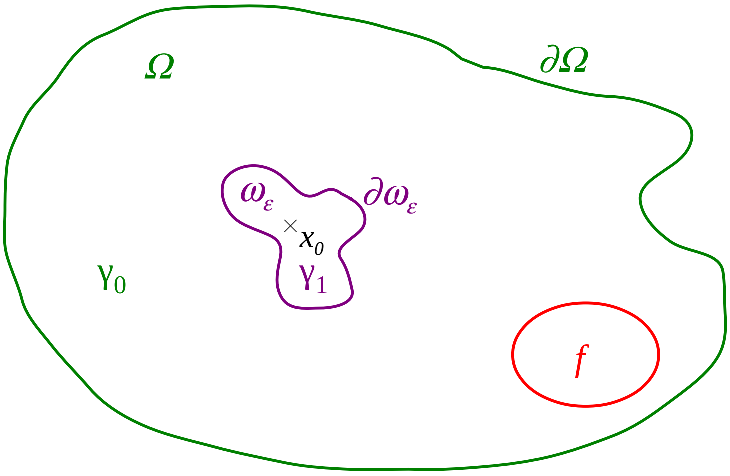

Figure 2.1: Perturbation of the conductivity in ω_ε.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$, $x_0 \in \Omega$ and a smooth bounded domain $\omega \subset \mathbb{R}^N$ such that $0 \in \omega$. For $\varepsilon > 0$, let $\omega_\varepsilon := x_0 + \varepsilon\, \omega$. For $\varepsilon > 0$ small enough it holds $\omega_\varepsilon \subset\subset \Omega$ and given two positive numbers $\gamma_0 \neq \gamma_1$ we define the perturbed conductivity by

$$\gamma_\varepsilon := \gamma_0 \ \text{ in } \Omega \setminus \omega_\varepsilon \text{ and } \gamma_\varepsilon := \gamma_1 \ \text{ in } \omega_\varepsilon.$$

Let a source $f \in L^2(\Omega)$ with a support $spt(f) \subset\subset \Omega \setminus \omega_\varepsilon$ for ε small enough. The perturbed direct state, denoted u_ε, is solution of the equation

$$\begin{cases} -\operatorname{div}(\gamma_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1.1}$$

The following steps are taken:

2.1.1 Variation of direct state defined in a Hilbert space.

Applying the Lax-Milgram theorem in the Hilbert space $\mathcal{H} = H_0^1(\Omega)$, u_ε is the unique solution in $\mathcal{H}$ of the following Euler-Lagrange equation :

find $u \in \mathcal{H}$ such that

$$\int_\Omega \gamma_\varepsilon \nabla u . \nabla \eta = \int_\Omega f \eta, \quad \forall \eta \in \mathcal{H}. \tag{2.1.2}$$

By linearity, the variation $\tilde{u}_\varepsilon := u_\varepsilon - u_0$ of the direct state is the unique solution of problem:
find $\tilde{u} \in \mathcal{H}$ such that

$$\int_\Omega \gamma_\varepsilon \nabla \tilde{u} . \nabla \eta + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} \nabla u_0 . \nabla \eta = 0, \quad \forall \eta \in \mathcal{H}. \tag{2.1.3}$$

2.1.2 Variation of adjoint state defined in the same Hilbert space.

Let $J : \mathcal{H} \to \mathbb{R}$ a Fréchet differentiable functional. Recall we aim at proving an asymptotic expansion of the form

$$J(u_\varepsilon) = J(u_0) + \rho(\varepsilon)\ g(x_0) + o(\rho(\varepsilon)), \tag{2.1.4}$$

where ρ is a nonnegative function such that $\lim_{\varepsilon \to 0} \rho(\varepsilon) = 0$.

The so-called adjoint state is by definition an element of the topological dual of $\mathcal{H}$. As usual the Hilbert space $\mathcal{H}$ is identified with its topological dual. The variational form defining the adjoint state v_ε is obtained considering the adjoint operator of equation (2.1.2) and $-DJ(u_0)$ as a source. In other words, applying again the Lax-Milgram theorem in $\mathcal{H}$, the perturbed adjoint state v_ε is defined as the unique solution of problem:
find $v \in \mathcal{H}$ such that

$$\int_\Omega \gamma_\varepsilon \nabla v . \nabla \eta = - \langle DJ(u_0), \eta \rangle, \quad \forall \eta \in \mathcal{H},$$

where $\langle . \, , \, . \rangle$ denotes the inner product in $\mathcal{H}$.

By linearity, the variation $\tilde{v}_\varepsilon := v_\varepsilon - v_0$ of the adjoint state is the unique solution of problem:
find $\tilde{v} \in \mathcal{H}$ such that

$$\int_\Omega \gamma_\varepsilon \nabla \tilde{v} . \nabla \eta + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} \nabla v_0 . \nabla \eta = 0, \quad \forall \eta \in \mathcal{H}. \tag{2.1.5}$$

Introducing the adjoint state, one can transform the first order Taylor expansion of functional J thanks to the fact that the variation $\tilde{u}_\varepsilon$ of the direct state belongs to the same Hilbert space as the adjoint state.

Thus plugging $\eta = \tilde{u}_\varepsilon$ in variational form (2.1.5), it follows

$$\begin{aligned} J(u_\varepsilon) - J(u_0) &= \langle DJ(u_0), \tilde{u}_\varepsilon \rangle + o\left(\|\tilde{u}_\varepsilon\|_{\mathcal{H}}\right) \\ &= - \int_\Omega \gamma_\varepsilon \nabla v_\varepsilon . \nabla \tilde{u}_\varepsilon + o\left(\|\tilde{u}_\varepsilon\|_{\mathcal{H}}\right) \\ &= - \int_\Omega \gamma_\varepsilon \nabla \tilde{v}_\varepsilon . \nabla \tilde{u}_\varepsilon - \int_\Omega \gamma_\varepsilon \nabla v_0 . \nabla \tilde{u}_\varepsilon + o\left(\|\tilde{u}_\varepsilon\|_{\mathcal{H}}\right). \end{aligned}$$

One also takes advantage of the fact that the unperturbed adjoint state v_0 belongs to the same Hilbert state as the direct state. Thus plugging $\eta = v_0$ in variational form (2.1.3), it follows

$$J(u_\varepsilon) - J(u_0) = -\int_\Omega \gamma_\varepsilon \nabla \tilde{v}_\varepsilon . \nabla \tilde{u}_\varepsilon \; + \; (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} \nabla u_0 . \nabla v_0 + o\left(\|\tilde{u}_\varepsilon\|_{\mathcal{H}}\right). \quad (2.1.6)$$

For simplicity, let us assume that J is a linear bounded form on $\mathcal{H}$, for instance the compliance

$$u \in \mathcal{H} \mapsto \int_\Omega f u.$$

Under this assumption, the remainder $o\left(\|\tilde{u}_\varepsilon\|_{\mathcal{H}}\right)$ in the Taylor expansion is zero.

Moreover, assuming enough regularity for $\partial\Omega$ and for the sources f and $DJ(u_0)$, we can assume (e.g. [46] Thm 8.34) that ∇u_0 and ∇v_0 are continuous in Ω and in particular at point x_0. Hence

$$\int_{\omega_\varepsilon} \nabla v_0 . \nabla u_0 = |\omega| \, \nabla v_0(x_0) . \nabla u_0(x_0) \; \varepsilon^N + o(\varepsilon^N).$$

Hence according to (2.1.6), the main task is to determine the asymptotic expansion of the following integral:

$$\int_\Omega \gamma_\varepsilon \nabla \tilde{v}_\varepsilon . \nabla \tilde{u}_\varepsilon. \quad (2.1.7)$$

2.1.3 Variations of direct state and of adjoint state at scale 1.

One thus introduces the variation H of direct state (resp. the variation K of adjoint state) at scale 1 in such a way that the following approximations hold

$$\tilde{u}_\varepsilon(x) \approx \varepsilon H(\varepsilon^{-1} x) \text{ and } \tilde{v}_\varepsilon(x) \approx \varepsilon K(\varepsilon^{-1} x), \quad \text{for a.e. } x \in \Omega.$$

Conductivity γ at scale 1 is defined by

$$\gamma := \gamma_0 \text{ in } \mathbb{R}^N \setminus \omega \quad \text{and} \quad \gamma_\varepsilon := \gamma_1 \text{ in } \omega. \quad (2.1.8)$$

An appropriate Hilbert space $\tilde{\mathcal{H}}$ of functions defined on $\mathbb{R}^N$ is then built ([40], volume 6, chapter XI and [8], Annexe A). A Poincaré inequality in $\tilde{\mathcal{H}}$ is required for coercivity. After (2.1.3) and applying the Lax-Milgram theorem in $\tilde{\mathcal{H}}$, one defines H as the unique solution of problem:
find $H \in \tilde{\mathcal{H}}$ such that

$$\int_{\mathbb{R}^N} \gamma \nabla H . \nabla \eta + (\gamma_1 - \gamma_0) \int_\omega \nabla u_0(x_0) . \nabla \eta = 0, \quad \forall \eta \in \tilde{\mathcal{H}}. \quad (2.1.9)$$

Similarly after (2.1.5) and identifying $\tilde{\mathcal{H}}$ with its topological dual, one defines K as the unique solution of problem:
find $K \in \tilde{\mathcal{H}}$ such that

$$\int_{\mathbb{R}^N} \gamma \nabla K . \nabla \eta + (\gamma_1 - \gamma_0) \int_\omega \nabla v_0(x_0) . \nabla \eta = 0, \quad \forall \eta \in \tilde{\mathcal{H}}. \quad (2.1.10)$$

Problem (2.1.10) defining K as problem (2.1.9) defining H, are two phases transmission problems in $\mathbb{R}^N$. Their sources are located on $\partial\omega$ and are of zero mean value. For instance the strong form of problem (2.1.10) is given by

$$\begin{cases} \Delta K = 0 & \text{in } \mathbb{R}^N \setminus \partial\omega, \\ \gamma_0 \left(\frac{\partial K}{\partial n_{out}}\right)_+ - \gamma_1 \left(\frac{\partial K}{\partial n_{out}}\right)_- = (\gamma_1 - \gamma_0)\nabla v_0(x_0).n_{out} & \text{on } \partial\omega. \end{cases} \tag{2.1.11}$$

where n_{out} denotes the unit outward normal to $\partial\omega$ and

$$\left(\frac{\partial K}{\partial n_{out}}\right)_\pm = \lim_{t\to 0^+} \nabla K(x \pm t n_{out}).n_{out}, \quad \forall x \in \partial\omega.$$

Such transmission problems were extensively studied (e.g. [5], Part I).

Once again, the integral

$$\int_{\mathbb{R}^N} \gamma \nabla K.\nabla H,$$

is well defined thanks to the duality that holds in Hilbert space $\tilde{\mathcal{H}}$ between the direct state H and the adjoint state K. Plugging the test function $K \in \tilde{\mathcal{H}}$ in variational form (2.1.9) and using the Green's formula, one obtains

$$-\int_{\mathbb{R}^N} \gamma \nabla K.\nabla H = (\gamma_1 - \gamma_0) \int_\omega \nabla u_0(x_0).\nabla K = (\gamma_1 - \gamma_0)\ \nabla u_0(x_0). \int_{\partial\omega} K n_{out},$$

where n_{out} denotes the unit outward normal to $\partial\omega$.

Regarding the calculation of the latter integral, it follows from the linearity of equation (2.1.11) defining K that the mapping

$$\nabla v_0(x_0) \mapsto (\gamma_1 - \gamma_0) \left[|\omega|\ \nabla v_0(x_0) + \int_{\partial\omega} K n_{out} \right]$$

is linear $\mathbb{R}^N \to \mathbb{R}^N$. It only depends on the set ω and on the ratio γ_1/γ_0. Thus there exists a N-dimensional matrix $\mathcal{P} = \mathcal{P}(\omega, \gamma_1/\gamma_0)$, called *polarization tensor*, such that

$$(\gamma_1 - \gamma_0) \left[|\omega|\ \nabla v_0(x_0) + \int_{\partial\omega} K n_{out} \right] \ = \ \mathcal{P} \nabla v_0(x_0)$$

(see e.g. [74, 34, 5, 12]). Such polarization tensor can be explicitly calculated for various types of sets ω, for instance for ellipsoids [52].

2.1.4 Asymptotic behavior of variations of direct and adjoint states at scale 1

The variational form (2.1.9) defining K may be rewritten

$$\int_{\mathbb{R}^N} \gamma_0 \nabla K.\nabla \eta + (\gamma_1 - \gamma_0) \int_\omega (\nabla K + \nabla v_0(x_0)).\nabla \eta = 0, \quad \forall \eta \in \tilde{\mathcal{H}}.$$

By convolution of the source with an elementary solution of the Laplace equation, one can estimate the asymptotic behavior of K and ∇K. It holds

$$K(y) = O\left(|y|^{1-N}\right) \text{ and } \nabla K(y) = O\left(|y|^{-N}\right) \text{ when } |y| \to +\infty.$$

Same asymptotic behavior holds for function H (resp. for gradient field ∇H).

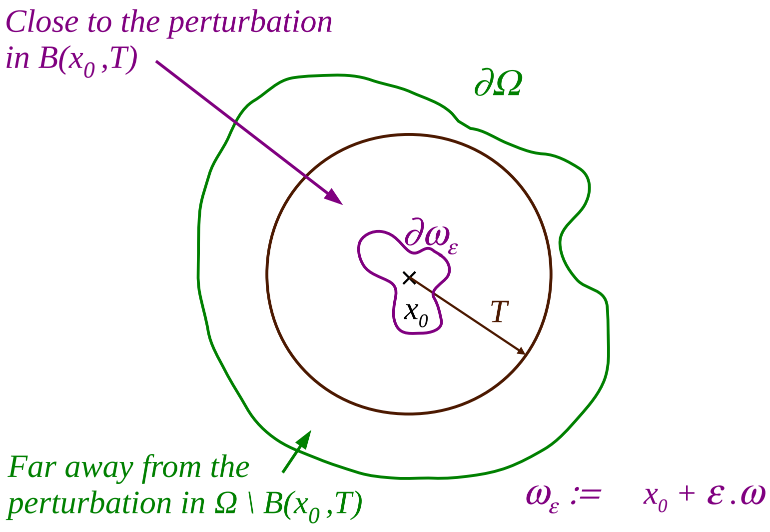

Figure 2.2: Close to the perturbation versus far away from the perturbation.

2.1.5 Estimation

The asymptotic behaviors of H and K and of their gradient fields play a key role in the estimations which yield

$$\int_\Omega \gamma_\varepsilon \nabla\tilde{v}_\varepsilon . \nabla\tilde{u}_\varepsilon = \varepsilon^N \int_{\mathbb{R}^N} \gamma \nabla K . \nabla H \ + \ o(\varepsilon^N).$$

It matters to know whether the variation of the direct state (resp. of the adjoint state) 'far away' from the perturbation ω_ε is of a higher order, i.e. is negligible, compared to the same variation 'near' the perturbation, as shown on Figure 2.2.

Such questions will have to be dealt with later on in Proposition 4.4.12, estimate (4.4.22) on page 63 and in Lemma 4.5.5, estimate (4.5.16) on page 66.

2.1.6 Conclusion

According to (2.1.6), it eventually follows the desired topological asymptotic expansion

$$J(u_\varepsilon) - J(u_0) = g(x_0)\ \varepsilon^N + \ o(\varepsilon^N),$$

with

$$g(x_0) = \nabla u_0(x_0).\left(\mathcal{P}\nabla v_0(x_0)\right) = \nabla u_0(x_0)^T \mathcal{P} \nabla v_0(x_0).$$

2.2 First steps taken for a quasilinear elliptic equation

As an example of quasilinear elliptic equation of second order, we choose to study the case of the p-Laplace equation.

Thus instead of considering the perturbation of the Laplace equation as in (2.1.1), we now consider the perturbation of the p-Laplace equation, $p \in (2, \infty)$. Put another way, the perturbed direct state u_ε satisfies the equation

$$\begin{cases} -\operatorname{div}\left(\gamma_\varepsilon \left|\nabla u\right|^{p-2} \nabla u\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^q(\Omega)$.

By comparison with the steps taken for the Laplace equation as described in the previous section, we are now going to emphasize the issue of defining the variation of direct state at scale 1 in $\mathbb{R}^N$.

In the space $\mathcal{V} = W_0^{1,p}(\Omega)$, it is standard ([60] or [17], §6.6) that u_ε is the unique minimizer in $\mathcal{V}$ of a Fréchet differentiable, strictly convex and coercive functional as follows

$$u_\varepsilon := \underset{u \in \mathcal{V}}{\operatorname{argmin}} \left\{ \int_\Omega \frac{\gamma_\varepsilon}{p} \left|\nabla u\right|^p - fu \right\},$$

and that equivalently function u_ε is defined as the unique solution to the Euler-Lagrange equation:
find $u \in \mathcal{V}$ such that

$$\int_\Omega \gamma_\varepsilon \left|\nabla u\right|^{p-2} \nabla u . \nabla \eta = \int_\Omega f \eta, \quad \forall \eta \in \mathcal{V}. \tag{2.2.1}$$

Denote again u_0 the unperturbed direct state and $\tilde{u}_\varepsilon := u_\varepsilon - u_0$ the variation of the direct state. Hence calculating the difference between equation (2.2.1) and the equation satisfied by u_0, one obtains that function $\tilde{u}_\varepsilon$ is solution of the following Euler-Lagrange equation:

$$\begin{aligned} &\text{find } \tilde{u} \in \mathcal{V} \text{ such that} \\ &\int_\Omega \gamma_\varepsilon \left[\left|\nabla u_0 + \nabla \tilde{u}\right|^{p-2} (\nabla u_0 + \nabla \tilde{u}) - \left|\nabla u_0\right|^{p-2} \nabla u_0 \right] . \nabla \eta \\ &\qquad\qquad + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} \left|\nabla u_0\right|^{p-2} \nabla u_0 . \nabla \eta = 0, \quad \forall \eta \in \mathcal{V}. \end{aligned} \tag{2.2.2}$$

Conversely consider equation (2.2.2) and the nonlinear operator $a_\varepsilon : \mathcal{V} \to \mathcal{V}^*$ defined by

$$\langle a_\varepsilon \tilde{u}, \eta \rangle := \int_\Omega \gamma_\varepsilon \left[\left|\nabla u_0 + \nabla \tilde{u}\right|^{p-2} (\nabla u_0 + \nabla \tilde{u}) - \left|\nabla u_0\right|^{p-2} \nabla u_0 \right] . \nabla \eta, \quad \forall \tilde{u}, \eta \in \mathcal{V}.$$

Due to Hölder's inequality, it is clear that a_ε is well defined. Moreover the Minty-Browder theorem ([30], Theorem 5.15) can be applied to operator a_ε so as to prove that equation (2.2.2) admits a unique solution in $\mathcal{V}$. Of course by uniqueness this solution equals $\tilde{u}_\varepsilon$.

We sketch briefly the arguments showing that a_ε satisfies the assumptions required by the Minty-Browder theorem, as similar arguments will be detailed later on to prove Proposition 4.4.4.

1. The continuity of a_ε is based on the following inequality ([60], p.73):

$$\begin{aligned}\left| |\varphi+\psi|^{p-2}(\varphi+\psi) - |\varphi|^{p-2}\varphi \right| &\le (p-1)\,|\psi| \int_0^1 |\varphi + t\psi|^{p-2}\, dt \\ &\le 2^{p-2}(p-1)\,|\psi| \left(|\varphi|^{p-2} + |\psi|^{p-2} \right), \quad \forall \varphi, \psi \in \mathbb{R}^N. \quad (2.2.3)\end{aligned}$$

   It follows from (2.2.3) and from Hölder's inequality that for all $u, v, \eta \in \mathcal{V}$,

$$\begin{aligned}|\langle a_\varepsilon(u+v) - a_\varepsilon(u), \eta\rangle| &\le C \int_\Omega \left[|\nabla(u_0+u)|^{p-2} |\nabla v| + |\nabla v|^{p-1} \right] |\nabla \eta| \\ &\le C \left[\|\nabla(u_0+u)\|_{L^p(\Omega)}^{p-2} \|\nabla v\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)}^{\frac{p}{q}} \right] \|\nabla \eta\|_{L^p(\Omega)},\end{aligned}$$

   with $C = 2^{p-2}(p-1)\max(\gamma_0, \gamma_1)$. Hence for all $u, v \in \mathcal{V}$, it holds

$$\|a_\varepsilon(u+v) - a_\varepsilon(u)\|_{\mathcal{V}^*} \le C \left[\|\nabla(u_0+u)\|_{L^p(\Omega)}^{p-2} \|\nabla v\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)}^{\frac{p}{q}} \right].$$

   The continuity of a_ε is proved.

2. As Ω is bounded, a Poincaré inequality holds in $\mathcal{V}$. Thus the norm $\|.\|_{\mathcal{V}}$ is equivalent to the semi-norm $|.|_{\mathcal{V}}$ in $\mathcal{V}$. The Poincaré inequality in $\mathcal{V}$ is essential to obtain strict monotony and coercivity of a_ε after applying the following p-ellipticity inequality ([60], page 71 (I)): for all $p \in (2, \infty)$, the exists $c = c(p) > 0$ such that

$$\left[|\varphi+\psi|^{p-2}(\varphi+\psi) - |\varphi|^{p-2}\varphi \right].\psi \ge c\ |\psi|^p, \quad \forall\ \varphi, \psi \in \mathbb{R}^N. \quad (2.2.4)$$

We now prepare the change of scale taking an intermediary step. Assume that u_0 is regular enough and denote $U_0 := \nabla u_0(x_0)$ its gradient at the center x_0 of perturbation. We approximate variation $\tilde{u}_\varepsilon$ by function $h_\varepsilon \in \mathcal{V}$ defined by Euler-Lagrange equation:

find $h \in \mathcal{V}$ such that

$$\begin{aligned}&\int_\Omega \gamma_\varepsilon \left[|U_0 + \nabla h|^{p-2}(U_0 + \nabla h) - |U_0|^{p-2} U_0 \right].\nabla \eta \\ &\qquad + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} |U_0|^{p-2} U_0.\nabla \eta = 0, \quad \forall \eta \in \mathcal{V}.\end{aligned}$$

Similarly the Minty-Browder theorem secures the existence and uniqueness of $h_\varepsilon \in \mathcal{V}$.

Moving now to scale 1, we look for a functional space $\mathcal{W}(\mathbb{R}^N)$ in which one can apply the Minty-Browder theorem to the following Euler-Lagrange equation:

find $H \in \mathcal{W}(\mathbb{R}^N)$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N} \gamma \left[|U_0 + \nabla H|^{p-2}(U_0 + \nabla H) - |U_0|^{p-2} U_0 \right].\nabla \eta \\ &\qquad + (\gamma_1 - \gamma_0) \int_{\omega} |U_0|^{p-2} U_0.\nabla \eta = 0, \quad \forall \eta \in \mathcal{W}(\mathbb{R}^N), \quad (2.2.5)\end{aligned}$$

where the perturbed conductivity γ at scale 1 is defined by (2.1.8).

For that purpose, we exclude the trivial case $U_0 = 0$. Thus assuming $U_0 \neq 0$, we have to find a space $\mathcal{W}(\mathbb{R}^N)$ such that the nonlinear operator $A : \mathcal{W}(\mathbb{R}^N) \to \mathcal{W}^*(\mathbb{R}^N)$ given by

$$\langle Au, \eta \rangle := \int_{\mathbb{R}^N} \gamma \left[|U_0 + \nabla u|^{p-2} (U_0 + \nabla u) - |U_0|^{p-2} U_0 \right] .\nabla \eta, \quad \forall u, \eta \in \mathcal{W}(\mathbb{R}^N),$$

is well defined in $\mathcal{W}(\mathbb{R}^N)$ and satisfies the assumptions required by the Minty-Browder theorem.

2.3 A two-norm discrepancy involving $L^p(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$

Studying the coercivity of A, one only has at disposal the p-ellipticity inequality (2.2.4) here above. Assume that the space $\mathcal{W}(\mathbb{R}^N)$ is built such that

$$\forall\ u \in \mathcal{W}(\mathbb{R}^N), \quad \nabla u \in L^p(\mathbb{R}^N).$$

Hence for all $u \in \mathcal{W}(\mathbb{R}^N)$, it holds

$$\langle Au, u \rangle = \int_{\mathbb{R}^N} \gamma \left[|U_0 + \nabla u|^{p-2} (U_0 + \nabla u) - |U_0|^{p-2} U_0 \right] .\nabla u \geq c' \left\| \nabla u \right\|^p_{L^p(\mathbb{R}^N)}, \tag{2.3.1}$$

with $c' = c \min(\gamma_0, \gamma_1) > 0$.

Therefore the coercivity of A could be secured in $\mathcal{W}(\mathbb{R}^N)$ should an equivalence hold in $\mathcal{W}(\mathbb{R}^N)$ between the norm $\|u\|_{\mathcal{W}(\mathbb{R}^N)}$ and the semi-norm $|u|_{\mathcal{W}(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)}$.

To obtain such an equivalence of the norm with the semi-norm (e.g. [40], volume 6, chapter XI and [8], Annexe A), one may think of building $\mathcal{W}(\mathbb{R}^N)$ as a quotient space

$$\mathcal{W}(\mathbb{R}^N) = \mathcal{W}^w(\mathbb{R}^N)/\mathbb{R}$$

where $\mathcal{W}^w(\mathbb{R}^N)$ is a weighted Sobolev space of the type

$$\mathcal{W}^w(\mathbb{R}^N) := \left\{ u \in \mathcal{D}'(\mathbb{R}^N);\ w_p u \in L^p(\mathbb{R}^N) \text{ and } \nabla u \in L^p(\mathbb{R}^N) \right\}$$

and where the weight $w_p : \mathbb{R}^N \to \mathbb{R}_+$ should be appropriately chosen. The weighted space $\mathcal{W}^w(\mathbb{R}^N)$ would be equipped with the norm

$$\|u\|_{\mathcal{W}^w(\mathbb{R}^N)} := \|w_p u\|_{L^p(\mathbb{R}^N)} \ + \ \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

The quotient space $\mathcal{W}(\mathbb{R}^N)$ would be equipped with the norm ([31], §11.2)

$$\|u\|_{\mathcal{W}(\mathbb{R}^N)} := \inf_{m \in \mathbb{R}} \|w_p(u+m)\|_{L^p(\mathbb{R}^N)} \ + \ \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

Assume that we have built such a quotient space and recall inequality (2.2.3). It follows that for all $u, \eta \in \mathcal{W}(\mathbb{R}^N)$

$$|\langle Au, \eta \rangle| \leq C \int_{\mathbb{R}^N} |\nabla u| \left(|U_0|^{p-2} + |\nabla u|^{p-2} \right) |\nabla \eta|$$

with $C = 2^{p-2}(p-1)\max(\gamma_0, \gamma_1)$. Thus A cannot be well defined in all the space $\mathcal{W}(\mathbb{R}^N)$. At least A is well defined in the subspace

$$\mathcal{V}(\mathbb{R}^N) := \Big\{u \in \mathcal{W}(\mathbb{R}^N); \nabla u \in L^2(\mathbb{R}^N)\Big\}$$

since for all $u, \eta \in \mathcal{V}(\mathbb{R}^N)$ the Schwarz's and Hölder's inequalities entail

$$|\langle Au, \eta\rangle| \leq C\left(|U_0|^{p-2} \|\nabla u\|_{L^2(\mathbb{R}^N)} \|\nabla \eta\|_{L^2(\mathbb{R}^N)} + \|\nabla u\|^{\frac{p}{q}}_{L^p(\mathbb{R}^N)} \|\nabla \eta\|_{L^p(\mathbb{R}^N)}\right). \quad (2.3.2)$$

So let us go further in this direction and endow the quotient space $\mathcal{V}(\mathbb{R}^N)$ with the norm

$$\|u\|_{\mathcal{V}(\mathbb{R}^N)} := \inf_{m\in\mathbb{R}} \|w_p(u+m)\|_{L^p(\mathbb{R}^N)} \ + \ \|\nabla u\|_{L^p(\mathbb{R}^N)} + \ \|\nabla u\|_{L^2(\mathbb{R}^N)}. \quad (2.3.3)$$

Then it follows from (2.3.2) that for all $u, \eta \in \mathcal{V}(\mathbb{R}^N)$, $\langle Au, \eta\rangle$ is well defined. Moreover for all $u \in \mathcal{V}(\mathbb{R}^N)$, Au is a bounded linear form on $\mathcal{V}(\mathbb{R}^N)$.

The fact is what has just been obtained on one side is lost on the other one. Inequality (2.3.1) shows that the term $\langle Au, u\rangle$ cannot provide control over the term $\|\nabla u\|_{L^2(\mathbb{R}^N)}$ of the norm $\|u\|_{\mathcal{V}(\mathbb{R}^N)}$. Therefore A is not coercive in $\mathcal{V}(\mathbb{R}^N)$ equipped with the norm (2.3.3).

2.4 Preliminary conclusion

In comparison with the method recalled in section 2.1, it thus appears that the step of defining the variation of the direct state at scale 1 by means of the Minty-Browder theorem, requires both

- to consider a functional space which, as described for $\mathcal{V}(\mathbb{R}^N)$, is equipped with a norm giving control on both the L^p and the L^2 norms of the gradient and which in addition enjoys a Poincaré inequality;
- to consider a quasilinear elliptic equation leading to an operator A enjoying both p- and 2- ellipticity properties, which is not the case for the p-Laplace equation.

The first requirement justifies that we build the reflexive Banach space $\mathcal{V}(\mathbb{R}^n)$ (and the Hilbert space $\mathcal{H}(\mathbb{R}^N)$ when $p = 2$) in the following chapter 3.

The second requirement explains why in chapter 4, on page 51, we consider a class of quasilinear equations for which operator A enjoys both p- and 2- ellipticity properties. The price to pay is we shall have to estimate quantities of the form

$$\|\nabla \tilde{u}\|^p_{L^p} + \|\nabla \tilde{u}\|^2_{L^2},$$

for the variation $\tilde{u}$ of the direct state at each step of approximation.

This situation may have some similarities with the two-norm discrepancies known in the field of nonlinear optimal control since the 1970's ([59, 51, 64, 4]). In such cases, control problems are differentiable in L^∞ norm but the second order condition holds in L^2 norm.

The present conclusion is only preliminary to the extent that, according to section 2.1 and on top of securing the definition of variation the direct state at scale 1, several other steps have to be implemented to obtain the topological asymptotic expansion. In particular in chapter 4, we shall have to:

1. ensure duality between the variation of the direct state, at each step of approximation, and the corresponding variation of the adjoint state. This task is straightforward in the linear case in the framework of Hilbert spaces. In chapter 4, we define the adjoint state and its variations as solutions of linearized equations in Hilbert spaces $\mathcal{H} = H_0^1(\Omega)$ and $\mathcal{H}(\mathbb{R}^n)$. The method we implement thus relies on the embedding $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$ at scale ε and, at scale 1, on the following 'fake embedding': for all $\eta \in L^\infty(\mathbb{R}^N)$, it holds

$$\eta \in \mathcal{V}(\mathbb{R}^n) \Rightarrow \eta \in \mathcal{H}(\mathbb{R}^n), \qquad \text{with} \qquad |u|_{\mathcal{H}(\mathbb{R}^n)} \leq |u|_{\mathcal{V}(\mathbb{R}^n)} .$$

   Otherwise one would have to call in more sophisticated functional frameworks, such as Gelfand triples ([94], §17).

2. determine the asymptotic behavior of the variation of the direct state at scale 1. This function will be solution of a transmission problem in $\mathcal{V}(\mathbb{R}^n)$, but of a nonlinear one for which techniques of convolution of source with elementary solution do not apply. We shall build a supersolution and a subsolution and then prove a comparison theorem.

3. determine with respect to the variation of the direct state, what does mean 'far away from the perturbation' ω_ε by opposition to 'close to the perturbation'. In the case of the class of quasilinear we choose, this question will be dealt with in Propositions 4.4.12 and 4.4.13.

Chapter 3

Weighted and quotient Sobolev spaces

The purpose of this chapter is to build an appropriate reflexive Banach space so as to define in chapter 4 the variation of the direct state at scale 1 in $\mathbb{R}^N$. In such a space, the variational form defining this variation has to comply with the requirements of the Minty-Browder theorem, that is to enjoy continuity, coercivity and strict monotony. Theorem 3.4.1 on page 46 provides the Poincaré inequality. Proposition 3.5.2 on page 49 secures the required coercivity property involving both the L^p and the L^2 norms of the gradient.

Similarly, we build an appropriate Hilbert space so as to define in chapter 4 the variation of the adjoint state at scale 1 in $\mathbb{R}^N$.

Hence this chapter extends to a nonlinear case in the framework of reflexive Banach spaces the definitions and theorems expounded in the linear case in the framework of Hilbert spaces in [40], volume 6, chapter XI, in [8], Annexe A and in [16], Appendix B. More general references about weighted Sobolev spaces can be found in [49] Chap.1, 15 and 20 and [88] Chap.1 and 2.

The building scheme of such spaces is classical. The main difficulty lies in the fact that in the nonlinear case $p \in (2, \infty)$, the quotient space cannot be identified with a closed subspace of the initial weighted Sobolev space. This issue is solved by Proposition 3.3.2 on page 45 which allows to complete the proof of the Poincaré inequality.

3.1 A Poincaré lemma

We define the weight function $w_p : \mathbb{R}^N \to \mathbb{R}$ as follows: for all $x \in \mathbb{R}^N$,

$$w_p(x) := \begin{cases} \left(1 + |x|^2\right)^{-\frac{N+1}{2p}} & \text{if } p < N, \\ \left(1 + |x|^2\right)^{-\frac{1}{2}} & \text{if } p > N, \\ \left(1 + |x|^2\right)^{-\frac{1}{2}} \left(\log(2 + |x|)\right)^{-1} & \text{if } p = N. \end{cases}$$

Note that

$$M_w := \sup_{x \in \mathbb{R}^N} w_p(x) < +\infty$$

and that

$$w_p(x) > 0, \quad \forall\, x \in \mathbb{R}^N \quad \text{with} \quad \inf_{x\in\mathbb{R}^N} w_p(x) = 0.$$

The exponents in w_p are chosen so that $w_p \in L^p(\mathbb{R}^N)$ and in particular $w_2 \in L^2(\mathbb{R}^N)$.

Recall that for all $a > 1$, we denote $B_a := \left\{x \in \mathbb{R}^N; |x| < a\right\}$ and $B'_a := \mathbb{R}^N \setminus \overline{B}_a$.

Lemma 3.1.1. *Let $a > 1$. Then there exists $c > 0$ such that for all $u \in C_0^\infty(B'_a)$ it holds*

$$\|w_p u\|_{L^p(B'_a)} \le c\, \|\nabla u\|_{L^p(B'_a)}\,.$$

Proof. Let $u \in C_0^\infty(B'_a)$. Let $\xi \in S^{N-1}$ and define $f \in C_0^\infty(a, +\infty)$ by

$$f(r) := u(r\xi) \quad \text{for all } r \in (a, +\infty).$$

We need to distinguish the cases $p \neq N$ and $p = N$.

1. Let us first consider the case $p \neq N$. It is easy to check that it holds

$$w_p(r) \le \frac{c(a)}{r}, \quad \forall\, r > a,$$

with $c(a) = a^{\frac{p-N-1}{p}}$ if $p < N$ and $c(a) = 1$ if $p > N$. It follows that

$$\int_a^\infty |w_p(r)f(r)|^p\, r^{N-1} dr \le c^p(a) \int_a^\infty |f(r)|^p\, r^{N-1-p}\mathrm{d}r.$$

Then it holds

$$\begin{aligned}
&\int_a^\infty |f(r)|^p\, r^{N-1-p}\mathrm{d}r \\
&= \frac{-p}{N-p} \int_a^\infty |f(r)|^{p-2}\, f(r)f'(r) r^{N-p}\mathrm{d}r \quad \text{integrating by parts} \\
&\le \frac{p}{|N-p|} \int_a^\infty \left[|f(r)|^{p-1}\, r^{N-p-\frac{N-1}{p}}\right] \left[|f'(r)|\, r^{\frac{N-1}{p}}\right] \mathrm{d}r \\
&\qquad \le \frac{p}{|N-p|} \left[\int_a^\infty |f(r)|^{q(p-1)}\, r^{q\left(N-p-\frac{N-1}{p}\right)}\mathrm{d}r\right]^{\frac{1}{q}} \left[\int_a^\infty |f'(r)|^p\, r^{N-1}\mathrm{d}r\right]^{\frac{1}{p}},
\end{aligned}$$

the last inequality being Hölder's inequality.

It holds $q(p-1) = p$ and

$$q\left(N - p - \frac{N-1}{p}\right) = q\left((N-1)\left(1 - \frac{1}{p}\right) + (1-p)\right) = N - 1 - p.$$

Hence

$$\int_a^\infty |w_p(r)f(r)|^p\, r^{N-1} dr \le c^p(a) \int_a^\infty |f(r)|^p\, r^{N-1-p}\mathrm{d}r \le c^p \int_a^\infty |f'(r)|^p\, r^{N-1}\mathrm{d}r \tag{3.1.1}$$

where $c := c(a)\; p/\,|N-p|$ does not depend on f.

2. Let us now consider the case $p = N$. It is easy to check that it holds

$$w_p^p(r)r^{N-1} \leq r^{-1}(\log r)^{-p}, \quad \forall\, r > a.$$

It follows that

$$\int_a^\infty |w_p(r)f(r)|^p\, r^{N-1}dr \leq \int_a^\infty |f(r)|^p\, \frac{1}{r(\log r)^p}\mathrm{d}r.$$

Then it holds

$$\begin{aligned}
&\int_a^\infty |f(r)|^p\, \frac{1}{r(\log r)^p}\mathrm{d}r \\
&= \frac{p}{p-1}\int_a^\infty |f(r)|^{p-2}\, f(r)f'(r)(\log r)^{1-p}\mathrm{d}r \quad \text{integrating by parts} \\
&\leq \frac{p}{p-1}\int_a^\infty \left[|f(r)|^{p-1}\, r^{\frac{1-p}{p}}(\log r)^{1-p}\right]\left[|f'(r)|\, r^{\frac{p-1}{p}}\right]\mathrm{d}r \\
&\qquad \leq \frac{p}{p-1}\left[\int_a^\infty |f(r)|^{q(p-1)}\, r^{\frac{q(1-p)}{p}}(\log r)^{q(1-p)}\right]^{\frac{1}{q}}\left[\int_a^\infty |f'(r)|^p\, r^{p-1}\right]^{\frac{1}{p}},
\end{aligned}$$

the last inequality being Hölder's inequality.
Since $q(p-1) = p$ and $q(1-p)/p = -1$ one obtains

$$\int_a^\infty |w_p(r)f(r)|^p\, r^{N-1}dr \leq \int_a^\infty |f(r)|^p\, \frac{1}{r(\log r)^p}\mathrm{d}r \leq c^p\int_a^\infty |f'(r)|^p\, r^{N-1}\mathrm{d}r, \tag{3.1.2}$$

where $c := p/(p-1)$ does not depend on f.

Then integrating (3.1.1) or (3.1.2) in spherical coordinates, it holds

$$\begin{aligned}
\int_{B'_a} |w_p(x)u(x)|^p\,\mathrm{d}x &= \int_{S^{N-1}}\int_a^\infty |w_p(r\xi)u(r\xi)|^p\, r^{N-1}\mathrm{d}r\mathrm{d}\xi \\
&= \int_{S^{N-1}}\left(\int_a^\infty |w_p(r)f(r)|^p\, r^{N-1}\mathrm{d}r\right)\mathrm{d}\xi \\
&\leq c^p\int_{S^{N-1}}\left(\int_a^\infty |f'(r)|^p\, r^{N-1}\mathrm{d}r\right)\mathrm{d}\xi \\
&= c^p\int_{S^{N-1}}\left(\int_a^\infty |\nabla u(r\xi).\xi|^p\, r^{N-1}\mathrm{d}r\right)\mathrm{d}\xi \\
&\leq c^p\int_{B'_a} |\nabla u(x)|^p\,\mathrm{d}x \quad \text{by Cauchy-Schwarz.}
\end{aligned}$$

Powering to $1/p$ one obtains the claimed inequality

$$\|w_p u\|_{L^p(B'_a)} \leq c\,\|\nabla u\|_{L^p(B'_a)}, \quad \forall u \in C_0^\infty(B'_a).$$

$\square$

3.2 Weighted Sobolev spaces

For all open subset $\mathcal{O} \subset \mathbb{R}^N$, recall we denote $\mathcal{D}'(\mathcal{O})$ the space of distributions in $\mathcal{O}$. Let the space

$$\mathcal{V}^w(\mathcal{O}) := \Big\{ u \in \mathcal{D}'(\mathcal{O}) \ ; \ w_p u \in L^p(\mathcal{O}), \nabla u \in L^p(\mathcal{O}) \cap L^2(\mathcal{O}) \Big\}$$

endowed with the norm defined by

$$\|u\|_{\mathcal{V}^w(\mathcal{O})} \ := \ \|w_p u\|_{L^p(\mathcal{O})} + \|\nabla u\|_{L^p(\mathcal{O})} + \|\nabla u\|_{L^2(\mathcal{O})}\,, \quad \forall u \in \mathcal{V}^w(\mathcal{O}).$$

For technical purposes it is also useful to define the larger space

$$\mathcal{W}^w(\mathcal{O}) := \{ u \in \mathcal{D}'(\mathcal{O}) \ ; \ w_p u \in L^p(\mathcal{O}), \nabla u \in L^p(\mathcal{O}) \}$$

endowed with the norm defined by

$$\|u\|_{\mathcal{W}^w(\mathcal{O})} \ := \ \|w_p u\|_{L^p(\mathcal{O})} + \|\nabla u\|_{L^p(\mathcal{O})}\,, \quad \forall u \in \mathcal{W}^w(\mathcal{O}).$$

Recall the usual Sobolev space

$$W^{1,p}(\mathcal{O}) := \{ u \in \mathcal{D}'(\mathcal{O}) \ ; u \in L^p(\mathcal{O}), \nabla u \in L^p(\mathcal{O}) \}$$

endowed with the norm defined by

$$\|u\|_{W^{1,p}(\mathcal{O})} \ := \ \|u\|_{L^p(\mathcal{O})} + \|\nabla u\|_{L^p(\mathcal{O})}\,, \quad \forall u \in W^{1,p}(\mathcal{O}).$$

Then we define the space

$$\mathcal{H}^w(\mathcal{O}) := \Big\{ u \in \mathcal{D}'(\mathcal{O}) \ ; \ w_2 u \in L^2(\mathcal{O}), \nabla u \in L^2(\mathcal{O}) \Big\}$$

endowed the inner product defined by

$$\langle u, v \rangle_{\mathcal{H}^w(\mathcal{O})} := \langle w_2 u, w_2 v \rangle_{L^2(\mathcal{O})} + \langle \nabla u, \nabla v \rangle_{L^2(\mathcal{O})}\,, \quad \forall u, v \in \mathcal{H}^w(\mathcal{O}).$$

and thus equipped with the norm

$$\|u\|_{\mathcal{H}^w(\mathcal{O})} := \langle u, u \rangle^{\frac{1}{2}}_{\mathcal{H}^w(\mathcal{O})}\,, \quad \forall u \in \mathcal{H}^w(\mathcal{O}).$$

When $p = 2$, the three normed spaces $\mathcal{V}^w(\mathcal{O})$, $\mathcal{W}^w(\mathcal{O})$ and $\mathcal{H}^w(\mathcal{O})$ are of course identical. Since we shall need to deal separately with the general case $p \in [2, \infty)$ and with the particular case $p = 2$, we distinguish $\mathcal{H}^w(\mathcal{O})$ from $\mathcal{V}^w(\mathcal{O})$ for the sake of clarity.

Lemma 3.2.1. *Assume that the open set $\mathcal{O}$ is bounded. Then the normed spaces $(\mathcal{V}^w(\mathcal{O}), \|.\|_{\mathcal{V}^w(\mathcal{O})})$, $(\mathcal{W}^w(\mathcal{O}), \|.\|_{\mathcal{W}^w(\mathcal{O})})$ and $(W^{1,p}(\mathcal{O}), \|.\|_{W^{1,p}(\mathcal{O})})$ are identical, in the algebraic sense and in the topological sense.*

Proof. Since w_p is positive and continuous in the compact $\overline{\mathcal{O}}$, there exist $m > 0$ such that $m \le w_p \le M_w$ in $\mathcal{O}$. Thus $\mathcal{W}^w(\mathcal{O}) = W^{1,p}(\mathcal{O})$ with equivalence of their norms. Moreover the embedding $L^p(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ is bounded. Thus $\mathcal{W}^w(\mathcal{O}) = \mathcal{V}^w(\mathcal{O})$ with equivalence of their norms. $\square$

Lemma 3.2.2. *The space $\mathcal{V}^w(\mathcal{O})$ endowed with the norm $\|.\|_{\mathcal{V}^w(\mathcal{O})}$ is a reflexive separable Banach space. The space $\mathcal{H}^w(\mathcal{O})$ endowed with the inner product $\langle .,. \rangle_{\mathcal{H}^w(\mathcal{O})}$ is a separable Hilbert space.*

Proof. Let us prove the assertion for $\mathcal{V}^w(\mathcal{O})$. Considering $p = 2$, the assertion will then immediately follow for $\mathcal{H}^w(\mathcal{O})$.

1. We first prove the completeness of $\mathcal{V}^w(\mathcal{O})$.
   Since $p \in [2, \infty)$, $L^p(\mathcal{O})$ and $L^2(\mathcal{O})$ are Banach spaces.
   Let$(u_l)_{l \in \mathbb{N}}$ be a Cauchy sequence of $\mathcal{V}^w(\mathcal{O})$. It follows from the definition of the norm $\|.\|_{\mathcal{V}^w(\mathcal{O})}$ that $(w_p u_l)$ is a Cauchy sequence in $L^p(\mathcal{O})$ and that (∇u_l) is a Cauchy sequence in both spaces $L^p(\mathcal{O})$ and $L^2(\mathcal{O})$. Therefore there exist $u, v \in L^p(\mathcal{O})$ and $w \in L^2(\mathcal{O})$ such that a subsequence of (u_l), still denoted (u_l) for simplicity, satisfies

$$\begin{cases} w_p u_l \to u & \text{in } L^p(\mathcal{O}), \\ \nabla u_l \to v & \text{in } L^p(\mathcal{O}), \\ \nabla u_l \to w & \text{in } L^2(\mathcal{O}). \end{cases}$$

   Let $\mathcal{O}_b$ be a bounded open subset of $\mathcal{O}$. Denote $\hat{u}_l$ (resp. $\hat{u}$, $\hat{v}$ and $\hat{w}$) the restriction of u_l (resp. u, v and w) to $\mathcal{O}_b$. It holds $w_p \hat{u}_l, \hat{u}, \nabla \hat{u}_l, \hat{v} \in L^p(\mathcal{O}_b)$ and $\nabla \hat{u}_l, \hat{w} \in L^2(\mathcal{O}_b)$ with

$$\begin{cases} w_p \hat{u}_l \to \hat{u} & \text{in } L^p(\mathcal{O}_b), \\ \nabla \hat{u}_l \to \hat{v} & \text{in } L^p(\mathcal{O}_b), \\ \nabla \hat{u}_l \to \hat{w} & \text{in } L^2(\mathcal{O}_b). \end{cases}$$

   Moreover, since $\mathcal{O}_b$ is bounded, the embedding $L^p(\mathcal{O}_b) \hookrightarrow L^2(\mathcal{O}_b)$ is bounded. It follows that

$$\nabla \hat{u}_l \to \hat{v} \quad \text{in } L^2(\mathcal{O}_b).$$

   Therefore the uniqueness of the limit in $L^2(\mathcal{O}_b)$ entails that $\hat{v} = \hat{w}$. Since this equality holds in all open bounded subset $\mathcal{O}_b \subset \mathcal{O}$, it follows

$$v = w \in L^p(\mathcal{O}) \cap L^2(\mathcal{O}).$$

   In addition, since $\mathcal{O}_b$ is bounded, there exists $m > 0$ such that $w_p \geq m$ in $\mathcal{O}_b$. Denote $\bar{u} := \hat{u}/w_p$. It holds

$$\int_{\mathcal{O}_b} |\bar{u}|^p \leq \frac{1}{m^p} \int_{\mathcal{O}_b} |w_p \bar{u}|^p \leq \frac{1}{m^p} \int_{\mathcal{O}} |u|^p < +\infty.$$

   Thus $\bar{u} \in L^p(\mathcal{O}_b)$. Similarly

$$\int_{\mathcal{O}_b} |\hat{u}_l - \bar{u}|^p \leq \frac{1}{m^p} \int_{\mathcal{O}_b} |w_p \hat{u}_l - \hat{u}|^p \leq \frac{1}{m^p} \int_{\mathcal{O}} |w_p u_l - u|^p .$$

   By assumption, the latter upper-bound satisfies

$$\lim_{l \to +\infty} \int_{\mathcal{O}} |w_p u_l - u|^p \ = \ 0.$$

Hence it holds simultaneously

$$\begin{cases} \hat{u}_l \to \bar{u} & \text{in } L^p(\mathcal{O}_b), \\ \nabla \hat{u}_l \to \hat{v} & \text{in } L^p(\mathcal{O}_b). \end{cases}$$

Thus $(\hat{u}_l)$ is a Cauchy sequence in the Banach space $W^{1,p}(\mathcal{O}_b)$. Due to the uniqueness of the limit in $W^{1,p}(\mathcal{O}_b)$, it holds $\nabla \bar{u} = \hat{v}$. Since this equality holds in all open bounded subset $\mathcal{O}_b \subset \mathcal{O}$, it follows that

$$\nabla(u/w_p) = v \in L^p(\mathcal{O}).$$

One can now conclude

$$\begin{cases} w_p u_l \to w_p(u/w_p) & \text{in } L^p(\mathcal{O}), \\ \nabla u_l \to \nabla(u/w_p) & \text{in } L^p(\mathcal{O}), \\ \nabla u_l \to \nabla(u/w_p) & \text{in } L^2(\mathcal{O}). \end{cases}$$

Hence the Cauchy sequence (u_l) converges to u/w_p in $\mathcal{V}^w(\mathcal{O})$. It follows that $\mathcal{V}^w(\mathcal{O})$ is a Banach space.

2. The proof of reflexivity and separability of $\mathcal{V}^w(\mathcal{O})$ can be obtained exactly as in the proof of Proposition 8.1 in [31], considering the product of reflexive separable spaces

$$E := L^p(\mathcal{O}) \times L^p(\mathcal{O}) \times L^2(\mathcal{O})$$

endowed with the norm

$$\|(u_1, u_2, u_3)\| := \|u_1\|_{L^p(\mathcal{O})} + \|u_2\|_{L^p(\mathcal{O})} + \|u_3\|_{L^2(\mathcal{O})}, \quad \forall\ (u_1, u_2, u_3) \in E.$$

□

Lemma 3.2.3. *The space $\mathcal{W}^w(\mathcal{O})$ endowed with the norm $\|.\|_{\mathcal{W}^w(\mathcal{O})}$ is a reflexive separable Banach space.*

Proof. The proof is a simplified version of that of Lemma 3.2.2, in which the L^2 norm of the gradient is not taken into account. □

Let $\mathcal{W}_0^w(\mathcal{O})$ be the closure of $C_0^\infty(\mathcal{O})$ in $\mathcal{W}^w(\mathcal{O})$ equipped with the norm $\|.\|_{\mathcal{W}^w(\mathcal{O})}$. As closed subspace of a reflexive separable Banach space, $\mathcal{W}_0^w(\mathcal{O})$ is still a reflexive separable Banach space.

Let us characterize $\mathcal{W}_0^w(\mathcal{O})$ in the case $\mathcal{O} = B'_a$ $(a > 0)$. Let $\theta : \mathbb{R}^N \to \mathbb{R}$ a smooth function such that $\theta = 1$ in B_{2a} and $\theta = 0$ in B'_{3a}. For all $u \in \mathcal{W}^w(\mathcal{O})$ it holds $u = \theta u + (1-\theta)u$ with $\theta u \in \mathcal{W}^w(B_{3a} \setminus B_a)$ and $(1-\theta)u \in \mathcal{W}^w(\mathbb{R}^N)$. Since $\mathcal{W}^w(B_{3a} \setminus B_a) = W^{1,p}(B_{3a} \setminus B_a)$, it is standard to define the trace of θu on ∂B_a, denoted $\gamma(\theta u)$. We define the trace of u on ∂B_a by $\gamma(u) := \gamma(\theta u)$.

If $u \in \mathcal{W}_0^w(B'_a)$, by definition there exists a sequence $(d_k) \subset C_0^\infty(B'_a)$ such that $d_k \to u$ in $\mathcal{W}^w(B'_a)$. Applying the Leibniz formula, it is straightforward that $\theta d_k \to \theta u$ in $W^{1,p}(B_{3a} \setminus B_a)$. Since $(\theta d_k) \subset C_0^\infty(B_{3a} \setminus B_a)$ it follows $\theta u \in W_0^{1,p}(B_{3a} \setminus B_a)$. Hence $\gamma(u) = \gamma(\theta u) = 0$ on ∂B_a.

Conversely, one may prove by standard techniques of regularization by mollifiers that $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathcal{W}^w(\mathbb{R}^N)$. Moreover as $(1-\theta)u = 0$ in B_{2a}, one may even build a sequence $(e_k) \subset C_0^\infty(\mathbb{R}^N)$ such that $e_k \to (1-\theta)u$ in $\mathcal{W}^w(\mathbb{R}^N)$ and such that for all $k \geq 0$, support$(e_k) \subset B'a$. If $\gamma(u) = \gamma(\theta u) = 0$ on ∂B_a, then there exists $(f_k) \subset C_0^\infty(B_{3a} \setminus B_a)$ such that $f_k \to \theta u$ in $W^{1,p}(B_{3a} \setminus B_a)$. Hence $e_k + f_k \to u$ in $\mathcal{W}^w(B'_a)$ with $(e_k + f_k) \subset C_0^\infty(B'_a)$. Therefore $u \in \mathcal{W}_0^w(B'_a)$.

One concludes

Lemma 3.2.4. *Let $a \geq 1$. It holds*

$$\mathcal{W}_0^w(B'_a) = \{u \in \mathcal{W}^w(B'_a); \gamma(u) = 0\}.$$

Next, applying Lemma 3.1.1 and due to the density of $C_0^\infty(B'_a)$ in $\mathcal{W}_0^w(B'_a)$, one obtains

Lemma 3.2.5. *Let $a > 1$. Then there exists $c > 0$ such that for all $u \in \mathcal{W}_0^w(B'_a)$ it holds*

$$\|w_p u\|_{L^p(B'_a)} \leq c \|\nabla u\|_{L^p(B'_a)}.$$

3.3 Weighted quotient Sobolev spaces

Recall that $w_p \in L^p(\mathbb{R}^N)$ and $w_2 \in L^2(\mathbb{R}^N)$. It follows that for all open set $\mathcal{O} \subset \mathbb{R}^N$, the space of constant functions defined on $\mathcal{O}$, identified with $\mathbb{R}$, is a subspace of $\mathcal{W}^w(\mathcal{O})$, of $\mathcal{V}^w(\mathcal{O})$ and of $\mathcal{H}^w(\mathcal{O})$.

Let $(m_n) \subset \mathbb{R}$ be a sequence of constant functions converging to $u \in \mathcal{W}^w(\mathcal{O})$. In particular it holds $\|\nabla m_n - \nabla u\|_{L^p(\mathcal{O})} = \|\nabla u\|_{L^p(\mathcal{O})} \to 0$ that is $\nabla u = 0$ a.e. . Hence $u \in \mathbb{R}$. Thus $\mathbb{R}$ is closed in $\mathcal{W}^w(\mathcal{O})$. Similarly $\mathbb{R}$ is closed in $\mathcal{V}^w(\mathcal{O})$ and as in $\mathcal{H}^w(\mathcal{O})$.

Then choosing $\mathcal{O} = \mathbb{R}^N$, we consider the quotient space $\mathcal{V}(\mathbb{R}^N) := \mathcal{V}^w(\mathbb{R}^N)/\mathbb{R}$ equipped with the norm

$$\|[u]\|_{\mathcal{V}(\mathbb{R}^N)} := \inf_{m \in \mathbb{R}} \|u + m\|_{\mathcal{V}^w(\mathbb{R}^N)}, \quad \forall\ [u] \in \mathcal{V}(\mathbb{R}^N) \tag{3.3.1}$$

where $u \in \mathcal{V}^w(\mathbb{R}^N)$ is any element of the class $[u]$. Since $\mathcal{V}^w(\mathbb{R}^N)$ is a reflexive Banach space and $\mathbb{R}$ is a closed subspace then $\mathcal{V}(\mathbb{R}^N)$ is still a reflexive Banach space (e.g. [31], chapter XI, §11.2).

Similarly we consider the quotient space $\mathcal{W}(\mathbb{R}^N) := \mathcal{W}^w(\mathbb{R}^N)/\mathbb{R}$ equipped with the norm

$$\|[u]\|_{\mathcal{W}(\mathbb{R}^N)} := \inf_{m \in \mathbb{R}} \|u + m\|_{\mathcal{W}^w(\mathbb{R}^N)}, \quad \forall\ [u] \in \mathcal{W}(\mathbb{R}^N) \tag{3.3.2}$$

where $u \in \mathcal{W}^w(\mathbb{R}^N)$ is any element of the class $[u]$. $\mathcal{W}(\mathbb{R}^N)$ is still a reflexive Banach space.

Then we define the quotient space $\mathcal{H}(\mathbb{R}^N) := \mathcal{H}^w(\mathbb{R}^N)/\mathbb{R}$ equipped with the norm

$$\|[u]\|_{\mathcal{H}(\mathbb{R}^N)} := \inf_{m \in \mathbb{R}} \|u + m\|_{\mathcal{H}^w(\mathbb{R}^N)}, \quad \forall\ [u] \in \mathcal{H}(\mathbb{R}^N) \tag{3.3.3}$$

where $u \in \mathcal{H}^w(\mathbb{R}^N)$ is any element of the class $[u]$. Since $\mathcal{H}^w(\mathbb{R}^N)$ is a Hilbert space and $\mathbb{R}$ is a closed subspace, then $\mathcal{H}(\mathbb{R}^N)$ is still a Hilbert space. Moreover the infimum in minimizing problem (3.3.3) is achieved at the unique point $m^* \in \mathbb{R}$ such that $-m^*$ is the orthogonal projection of u on the subspace $\mathbb{R}$ in the sense of the inner product $\langle .,. \rangle_{\mathcal{H}^w(\mathbb{R}^N)}$. Equivalently m^* is uniquely determined by

$$m^* \in \mathbb{R} \;\text{ and }\; \langle u + m^*, 1 \rangle_{\mathcal{H}^w(\mathbb{R}^N)} = 0. \tag{3.3.4}$$

Hence the norm $\|.\|_{\mathcal{H}(\mathbb{R}^N)}$ in $\mathcal{H}(\mathbb{R}^N)$ is indeed induced by the inner product defined by

$$\langle [u_1], [u_2] \rangle_{\mathcal{H}(\mathbb{R}^N)} := \langle u_1 + m_1^*, u_2 + m_2^* \rangle_{\mathcal{H}^w(\mathbb{R}^N)} \quad \forall\, [u_1], [u_2] \in \mathcal{H}(\mathbb{R}^N)$$

where, for $i = 1, 2$, u_i is any element of the class $[u_i]$ and m_i^* is determined by equation (3.3.4).

Generalizing (3.3.4) to the nonlinear case $p \in [2, +\infty)$, it holds

Lemma 3.3.1. *Let $[u] \in \mathcal{W}(\mathbb{R}^N)$ and let $u \in \mathcal{W}^w(\mathbb{R}^N)$ an element of the class $[u]$. Then the minimization problem (3.3.2) admits a unique solution $m^* \in \mathbb{R}$. The minimizing solution $v^* := u + m^* \in \mathcal{W}^w(\mathbb{R}^N)$ is uniquely determined by the following Euler-Lagrange equation:*

$$v^* \in u + \mathbb{R} \quad \textit{and} \quad \int_{\mathbb{R}^N} w_p^p \, |v^*|^{p-2} \, v^* = 0. \tag{3.3.5}$$

Assume in addition that $[u] \in \mathcal{V}(\mathbb{R}^N)$. Then m^* is the unique minimizer of problem (3.3.1) and it holds

$$\|[u]\|_{\mathcal{V}(\mathbb{R}^N)} = \|v^*\|_{\mathcal{V}^w(\mathbb{R}^N)} \,.$$

Proof. For all $v \in \mathcal{W}^w(\mathbb{R}^N)$ it follows from Hölder's inequality that

$$\int_{\mathbb{R}^N} w_p^p \, |v|^{p-1} \le \|w_p v\|_{L^p(\mathbb{R}^N)}^{\frac{p}{q}} \, \|w_p\|_{L^p(\mathbb{R}^N)} < +\infty.$$

Thus the integral quoted in (3.3.5) is well defined.

Then let $[u] \in \mathcal{W}(\mathbb{R}^N)$ and let $u \in \mathcal{W}^w(\mathbb{R}^N)$ an element of the class $[u]$. Since $\nabla m = 0$ for all $m \in \mathbb{R}$ and since $\int_{\mathbb{R}^N} |\nabla u|^p$ is a constant which does not depend on m, minimizing problem (3.3.1) is equivalent to minimizing the function

$$\begin{aligned} f \;:\; & \mathbb{R} \to \mathbb{R} \\ & m \mapsto \int_{\mathbb{R}^N} w_p^p \, |u + m|^p \,. \end{aligned} \tag{3.3.6}$$

1. Applying the Lebesgue dominated convergence theorem, the differentiability of f in $\mathbb{R}$ is standard. It holds

$$f'(m) = p \int_{\mathbb{R}^N} w_p^p \, |u + m|^{p-2} \, (u + m), \quad \forall m \in \mathbb{R}.$$

2. The strict convexity of the function $m \in \mathbb{R} \mapsto |m|^p$ entails that of f.

3. Again the convexity of $m \in \mathbb{R} \mapsto |m|^p$ implies that

$$\left|\frac{1}{2} m_1\right|^p \leq \frac{1}{2} |-m_2|^p + \frac{1}{2} |m_1 + m_2|^p, \quad \forall m_1, m_2 \in \mathbb{R}.$$

It follows

$$f(m) \geq \frac{|m|^p}{2^{p-1}} \int_{\mathbb{R}^N} w_p^p \; - \; \int_{\mathbb{R}^N} w_p^p |u|^p,$$

which proves the coercivity of f.

Since f is differentiable, strictly convex and coercive in $\mathbb{R}$, it admits a unique minimizer m^* which is uniquely determined by the first order condition $f'(m^*) = 0$.
Thus $v^* = u + m^*$ is uniquely determined by the Euler-Lagrange condition (3.3.5).

Lastly assume that in addition $[u] \in \mathcal{V}(\mathbb{R}^N)$. By definition it holds

$$\|u + m\|_{\mathcal{V}^w(\mathbb{R}^N)} = \|u + m\|_{\mathcal{W}^w(\mathbb{R}^N)} + \|\nabla u\|_{L^2(\mathcal{O})}, \quad \forall m \in \mathbb{R}.$$

Therefore minimizing problem (3.3.1) is again equivalent to minimizing function f defined by (3.3.6) in $\mathbb{R}$. Thus m^* is the unique minimizer in $\mathbb{R}$ of problem (3.3.1). It follows that

$$\|[u]\|_{\mathcal{V}(\mathbb{R}^N)} = \|v^*\|_{\mathcal{V}^w(\mathbb{R}^N)}.$$

□

Let us study the set of functions $v \in \mathcal{W}^w(\mathbb{R}^N)$ minimizing problem (3.3.2) when u ranges over $\mathcal{W}(\mathbb{R}^N)$. According to Lemma 3.3.1, such minimizing functions satisfy the Euler-Lagrange equation 3.3.5.

Conversely, let $v \in \mathcal{W}^w(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} w_p^p |v|^{p-2} v = 0.$$

Then v is obviously the minimizer of problem (3.3.2) for the class $[v] \in \mathcal{W}(\mathbb{R}^N)$.

Hence the set

$$\mathcal{M}_p := \left\{ v \in \mathcal{W}^w(\mathbb{R}^N); \int_{\mathbb{R}^N} w_p^p |v|^{p-2} v = 0 \right\} \tag{3.3.7}$$

is the set of minimizing functions of problem (3.3.2) when $[u]$ ranges over the space $\mathcal{W}(\mathbb{R}^N)$.

In the linear case $p = 2$, the set $\mathcal{M}_2$ is the closed vector subspace $\mathbb{R}^\perp$ in the Hilbert space $\mathcal{H}(\mathbb{R}^N)$ and the proof of the Poincaré inequality is eventually completed noticing that $\mathbb{R}^\perp \cap \mathbb{R} = \{0\}$. When $p > 2$, it is easy to see that $\mathcal{M}_p$ does not satisfy the additivity condition ($v_1, v_2 \in \mathcal{M}_p \not\Rightarrow v_1 + v_2 \in \mathcal{M}_p$). $\mathcal{M}_p$ enjoys the structure of a cone ($0 \in \mathcal{M}_p$ and $v \in \mathcal{M}_p \Rightarrow \lambda v \in \mathcal{M}_p$, for all $\lambda \in \mathbb{R}$).

As a generalization to the property $\mathbb{R}^\perp \cap \mathbb{R} = \{0\}$ applied in the Hilbertian case $p = 2$, it holds

Proposition 3.3.2. *Let $(v_l)_{l \in \mathbb{N}}$ a sequence $\subset \mathcal{M}_p$ and $m \in \mathbb{R}$ such that*

$$\lim_{l \to +\infty} \int_{\mathbb{R}^N} w_p^p |v_l + m|^{p-2} (v_l + m) \; = \; 0. \tag{3.3.8}$$

Then $m = 0$.

We first recall an inequality from [60], page 73. Let $p \geq 2$ and a dimension $d \in \mathbb{N}^*$. Then there exists $c_p > 0$ such that

$$\left\langle |x_2|^{p-2}\, x_2 - |x_1|^{p-2}\, x_1, x_2 - x_1 \right\rangle \geq c_p\, |x_2 - x_1|^p\,, \quad \forall x_1, x_2 \in \mathbb{R}^d, \tag{3.3.9}$$

where $\langle .,. \rangle$ denotes the usual euclidean inner product in $\mathbb{R}^d$. We apply this inequality hereafter in the case $d = 1$.

Proof. By assumption, for all $l \in \mathbb{N}$, $v_l \in \mathcal{M}_p$. Hence the Euler-Lagrange equation reads

$$\int_{\mathbb{R}^N} w_p^p\, |v_l|^{p-2}\, v_l = 0, \quad \forall\ l \in \mathbb{N}.$$

Calculating the difference of the latter with (3.3.8) we obtain

$$\lim_{l \to +\infty} \int_{\mathbb{R}^N} w_p^p \left[|v_l + m|^{p-2}\, (v_l + m) - |v_l|^{p-2}\, v_l \right] \;=\; 0. \tag{3.3.10}$$

After inequality (3.3.9) it holds for a.e. $x \in \mathbb{R}^N$

$$\left[|v_l(x) + m|^{p-2}\, (v_l(x) + m) - |v_l(x)|^{p-2}\, v_l \right] m \geq c_p\, |m|^p \geq 0.$$

Hence it follows from (3.3.10) that

$$\lim_{l \to +\infty} \int_{\mathbb{R}^N} w_p^p\ c_p\ \ |m|^p \;=\; 0.$$

Therefore $m = 0$. □

In geometric terms, condition (3.3.8) says that $v_l + m$ asymptotically becomes a minimizer for the class $[v_l + m] = [v_l]$. Since $v_l \in \mathcal{M}_p$ is minimizer for this class, the uniqueness of the minimizer roughly speaking leads to $m = 0$.

3.4 Poincaré inequality in quotient Sobolev space $\mathcal{W}(\mathbb{R}^N)$

We now come to the main results of this chapter.

Theorem 3.4.1. *There exists $c > 0$ such that*

$$\|[u]\|_{\mathcal{W}(\mathbb{R}^N)} \leq c\, \|\nabla u\|_{L^p(\mathbb{R}^N)}\,, \quad \forall\ [u] \in \mathcal{W}(\mathbb{R}^N).$$

where $u \in \mathcal{W}^w(\mathbb{R}^N)$ is any element of the class $[u]$.

Proof. Assume that the Theorem does not hold. Then there exists a sequence

$$([u_l])_{l \in \mathbb{N}} \subset \mathcal{W}(\mathbb{R}^N)$$

such that

$$\|[u_l]\|_{\mathcal{W}(\mathbb{R}^N)} = 1 \;\text{ and }\; \|\nabla u_l\|_{L^p(\mathbb{R}^N)} \leq 1/l, \quad \forall l \geq 1, \tag{3.4.1}$$

where u_l denotes any element of the class $[u_l]$.

For all $l \geq 1$, denote $v_l \in \mathcal{M}_p$ the element of $\mathcal{W}^w(\mathbb{R}^N)$ which minimizes problem (3.3.1) for the class $[u_l]$. One can rewrite (3.4.1) as follows:

$$\forall\, l \geq 1, \quad \text{it holds} \quad v_l \in \mathcal{M}_p, \tag{3.4.2}$$

$$\|v_l\|_{\mathcal{W}^w(\mathbb{R}^N)} = 1, \tag{3.4.3}$$

$$\|\nabla v_l\|_{L^p(\mathbb{R}^N)} \leq 1/l. \tag{3.4.4}$$

Let $a > 1$. Let $\theta : \mathbb{R}^N \to \mathbb{R}$ a smooth function such that $\theta = 1$ in B_a and $\theta = 0$ in $B_{2a}^{'}$. Denote $c_\theta := \sup\left\{\max(|\theta(x)|, |1-\theta(x)|, |\nabla\theta(x)|); x \in \mathbb{R}^N\right\} < +\infty$.

For all $v \in \mathcal{W}^w(\mathbb{R}^N)$, it holds $v = \theta v + (1-\theta)v$ with $\theta v \in \mathcal{W}^w(B_{2a})$ and $(1-\theta)v \in \mathcal{W}^w(B_a^{'})$.

1. First consider what happens in $\mathcal{W}_0^w(B_{2a})$.
   According to Lemma 3.2.1, it holds $\mathcal{W}^w(B_{2a}) = W^{1,p}(B_{2a})$ with equivalence of the norms. Thus $\mathcal{W}_0^w(B_{2a}) = W_0^{1,p}(B_{2a})$. Moreover it holds $\theta v \in W_0^{1,p}(B_{2a})$ for all $v \in \mathcal{W}^w(\mathbb{R}^N)$.
   Since
   $$\|v\|_{\mathcal{W}^w(B_{2a})} \leq \|v\|_{\mathcal{W}^w(\mathbb{R}^N)}, \quad \forall v \in \mathcal{W}^w(\mathbb{R}^N),$$
   it follows from (3.4.3) that the sequence $(v_l)_l$ is bounded in $W^{1,p}(B_{2a})$. According to the Rellich-Kondrakov theorem, the embedding $W^{1,p}(B_{2a}) \hookrightarrow L^p(B_{2a})$ is compact. Hence there exists a subsequence of (v_l), which we still denote (v_l) for simplicity, such that (v_l) converges in $L^p(B_{2a})$.
   Moreover the Leibniz formula reads
   $$\nabla(\theta v) = (\nabla\theta)v + \theta(\nabla v), \quad \forall\, v \in \mathcal{W}^w(\mathbb{R}^N).$$
   Thus
   $$\|\nabla(\theta v)\|_{L^p(B_{2a})} \leq c_\theta \left(\|v\|_{L^p(B_{2a})} + \|\nabla v\|_{L^p(B_{2a})}\right), \; \forall v \in \mathcal{W}^w(\mathbb{R}^N). \tag{3.4.5}$$
   In addition, according to the Poincaré inequality in $W_0^{1,p}(B_{2a})$ (see e.g. [17] Th. 5.3.1), there exists $C_{2a} > 0$ such that
   $$\|\theta v\|_{W^{1,p}(B_{2a})} \leq C_{2a} \|\nabla(\theta v)\|_{L^p(B_{2a})}, \quad \forall\, v \in \mathcal{W}^w(\mathbb{R}^N). \tag{3.4.6}$$
   Therefore for all $k, l \geq 1$ it holds
   $$\begin{aligned}
   &\frac{1}{C_{2a}} \|\theta(v_k - v_l)\|_{W^{1,p}(B_{2a})} \leq \|\nabla(\theta(v_k - v_l))\|_{L^p(B_{2a})} \text{ after (3.4.6)} \\
   &\leq c_\theta \left(\|v_k - v_l\|_{L^p(B_{2a})} + \|\nabla(v_k - v_l)\|_{L^p(B_{2a})}\right) \text{ after (3.4.5)} \\
   &\leq c_\theta \left(\|v_k - v_l\|_{L^p(B_{2a})} + \|\nabla v_k\|_{L^p(B_{2a})} + \|\nabla v_l\|_{L^p(B_{2a})}\right) \\
   &\qquad\qquad \leq c_\theta \left(\|v_k - v_l\|_{L^p(B_{2a})} + 1/k + 1/l\right) \text{ after (3.4.4).}
   \end{aligned}$$
   Since $(v_l)_l$ is converging in $L^p(B_{2a})$, it follows that $(\theta v_l)_l$ is a Cauchy sequence in the Banach space $W^{1,p}(B_{2a})$. Hence there exists $v_{2a} \in W^{1,p}(B_{2a})$ such that $\theta v_l \to v_{2a}$ in $W^{1,p}(B_{2a})$.

2. Then consider what happens in $\mathcal{W}_0^w(B'_a)$. After Lemma 3.2.4, for all $v \in \mathcal{W}^w(\mathbb{R}^N)$ it holds $(1-\theta)v \in \mathcal{W}_0^w(B'_a)$.
   The Leibniz formula reads
   $$\nabla((1-\theta)v) = -(\nabla\theta)v + (1-\theta)(\nabla v), \quad \forall\, v \in \mathcal{W}^w(\mathbb{R}^N).$$
   Since
   $$\nabla\theta = 0 \quad \text{in } B_a \cup B'_{2a},$$
   it follows
   $$\|\nabla\left[(1-\theta)v\right]\|_{L^p(B'_a)} \le c_\theta \left(\|v\|_{L^p(B_{2a}\setminus B_a)} + \|\nabla v\|_{L^p(B'_a)}\right),\ \forall\, v \in \mathcal{W}^w(\mathbb{R}^N). \tag{3.4.7}$$
   Let $m := \inf\{w_p(x); x \in B_{2a}\} > 0$. Since
   $$\|v\|_{W^{1,p}(B_{2a}\setminus B_a)} \le \max(1, 1/m)\ \|v\|_{\mathcal{W}^w(\mathbb{R}^N)}, \quad \forall v \in \mathcal{W}^w(\mathbb{R}^N),$$
   it follows from (3.4.3) that the sequence (v_l) is bounded in $W^{1,p}(B_{2a} \setminus B_a)$.
   According to the Rellich-Kondrakov theorem, the embedding $W^{1,p}(B_{2a} \setminus B_a) \hookrightarrow L^p(B_{2a} \setminus B_a)$ is compact. Hence there exists a subsequence of (v_l), which we still denote (v_l) for simplicity, such that (v_l) converges in $L^p(B_{2a} \setminus B_a)$.
   Hence for all $l \ge 1$ it holds
   $$\begin{aligned}
   &\|(1-\theta)(v_k - v_l)\|_{\mathcal{W}^w(B'_a)} \\
   &= \|w_p(1-\theta)(v_k - v_l)\|_{L^p(B'_a)} + \|\nabla\left[(1-\theta)(v_k - v_l)\right]\|_{L^p(B'_a)} \\
   &\le (c+1)\,\|\nabla\left[(1-\theta)(v_k - v_l)\right]\|_{L^p(B'_a)} \text{ after Poincaré Lemma 3.2.5,} \\
   &\le (c+1)c_\theta\left(\|v_k - v_l\|_{L^p(B_{2a}\setminus B_a)} + \|\nabla(v_k - v_l)\|_{L^p(B'_a)}\right) \text{ after (3.4.7),} \\
   &\qquad\qquad \le (c+1)c_\theta\left(\|v_k - v_l\|_{L^p(B_{2a}\setminus B_a)} + 1/k + 1/l\right) \text{ after (3.4.4).}
   \end{aligned}$$
   Therefore $((1-\theta)v_l)_l$ is a Cauchy sequence in $\mathcal{W}^w(B'_a)$. Hence there exists $v_{a'} \in \mathcal{W}^w(B'_a)$ such that $(1-\theta)v_l \to v_{a'}$ in $\mathcal{W}^w(B'_a)$.

Denote $v_\infty := v_{2a} + v_{a'} \in \mathcal{W}^w(\mathbb{R}^N)$. By summation, it follows that $v_l \to v_\infty$ in $\mathcal{W}^w(\mathbb{R}^N)$.

In particular it holds $\nabla v_l \to \nabla v_\infty$ in $L^p(\mathbb{R}^N)$. By assumption $\|\nabla v_l\|_{L^p(\mathbb{R}^N)} \to 0$. Hence $\|\nabla v_\infty\|_{L^p(\mathbb{R}^N)} = 0$ which entails that v_∞ is a constant function in $\mathbb{R}^N$. Moreover after Hölder's inequality, it holds
$$\begin{aligned}
\left|\int_{\mathbb{R}^N} w_p^p\,|v_l - v_\infty|^{p-2}\,(v_l - v_\infty)\right| \;&\le\; \|w_p\|_{L^p(\mathbb{R}^N)}\,\|w_p(v_l - v_\infty)\|_{L^p(\mathbb{R}^N)}^{\frac{p}{q}} \\
&\le\; \|w_p\|_{L^p(\mathbb{R}^N)}\,\|v_l - v_\infty\|_{\mathcal{W}^w(\mathbb{R}^N)}^{\frac{p}{q}}.
\end{aligned}$$
Since $v_l \to v_\infty$ in $\mathcal{W}^w(\mathbb{R}^N)$, it follows that
$$\lim_{l\to+\infty}\int_{\mathbb{R}^N} w_p^p\,|v_l - v_\infty|^{p-2}\,(v_l - v_\infty) \;=\; 0.$$
Hence one can apply Proposition 3.3.2 to the sequence $(v_l) \subset \mathcal{M}_p$ and to the constant $-v_\infty$. It follows that $v_\infty = 0$. Therefore $v_l \to 0$ in $\mathcal{W}^w(\mathbb{R}^N)$ which contradicts the assumption $\|v_l\|_{\mathcal{W}^w(\mathbb{R}^N)} = 1$, for all $l \ge 1$. □

For all $[u] \in \mathcal{W}(\mathbb{R}^N)$, let $u \in \mathcal{W}^w(\mathbb{R}^N)$ any element in the class $[u]$. Endow $\mathcal{W}(\mathbb{R}^N)$ with the semi-norm

$$|[u]|_{\mathcal{W}(\mathbb{R}^N)} := \|\nabla u\|_{L^p(\mathbb{R}^N)} . \tag{3.4.8}$$

It follows immediately from Poincaré Theorem 3.4.1 that

Corollary 3.4.2. *The semi-norm* $| \, . \, |_{\mathcal{W}(\mathbb{R}^N)}$ *and the norm* $\| \, . \, \|_{\mathcal{W}(\mathbb{R}^N)}$ *are equivalent in* $\mathcal{W}(\mathbb{R}^N)$.

3.5 Poincaré inequality & coercivity in quotient space $\mathcal{V}(\mathbb{R}^N)$

Let $[u] \in \mathcal{V}(\mathbb{R}^N)$ and $u \in \mathcal{V}^w(\mathbb{R}^N)$ any element of the class $[u]$. Endow $\mathcal{V}(\mathbb{R}^N)$ with the semi-norm given by

$$|[u]|_{\mathcal{V}(\mathbb{R}^N)} := \|\nabla u\|_{L^p(\mathbb{R}^N)} + \|\nabla u\|_{L^2(\mathbb{R}^N)} . \tag{3.5.1}$$

Again it follows immediately from Poincaré Theorem 3.4.1 that

Corollary 3.5.1. *The semi-norm* $| \, . \, |_{\mathcal{V}(\mathbb{R}^N)}$ *and the norm* $\| \, . \, \|_{\mathcal{V}(\mathbb{R}^N)}$ *are equivalent in* $\mathcal{V}(\mathbb{R}^N)$.

We can now state the expected combined p- and 2- coercivity property which is the second main result of this chapter.

Proposition 3.5.2. *For all* $[u] \in \mathcal{V}(\mathbb{R}^N)$, *denote* $u \in \mathcal{V}^w(\mathbb{R}^N)$ *any element in the class* $[u]$. *Then it holds*

$$\lim_{\|[u]\|_{\mathcal{V}(\mathbb{R}^N)} \to \infty} \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{\|[u]\|_{\mathcal{V}(\mathbb{R}^N)}} = +\infty.$$

Proof. To study the limit at infinity, one can assume that

$$|[u]|_{\mathcal{V}(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} + \|\nabla u\|_{L^2(\mathbb{R}^N)} \geq 1.$$

1. If $\|\nabla u\|_{L^p(\mathbb{R}^N)} \leq 1$, then it holds
$$\begin{aligned} \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} &\geq \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} \\ &\geq \frac{(|[u]|_{\mathcal{V}(\mathbb{R}^N)} - 1)^2}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}}. \end{aligned}$$

2. If $\|\nabla u\|_{L^p(\mathbb{R}^N)} > 1$, since $2 \leq p < \infty$, then it holds
$$\begin{aligned} \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} &\geq \frac{\|\nabla u\|^2_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} \\ &\geq \frac{1}{2} |[u]|_{\mathcal{V}(\mathbb{R}^N)} . \end{aligned}$$

Thus for all $[u] \in \mathcal{V}(\mathbb{R}^N)$ such that $|[u]|_{\mathcal{V}(\mathbb{R}^N)} \geq 1$, it holds

$$\frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} \;\geq\; \min\left(\frac{(|[u]|_{\mathcal{V}(\mathbb{R}^N)} - 1)^2}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}}, \frac{|[u]|_{\mathcal{V}(\mathbb{R}^N)}}{2}\right).$$

Hence

$$\lim_{|[u]|_{\mathcal{V}(\mathbb{R}^N)}\to\infty} \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} \;=\; +\infty.$$

Thus after the equivalence stated in Corollary 3.5.1, we obtain the claimed limit

$$\lim_{\|[u]\|_{\mathcal{V}(\mathbb{R}^N)}\to\infty} \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{\|[u]\|_{\mathcal{V}(\mathbb{R}^N)}} \;=\; +\infty.$$

□

3.6 Poincaré inequality and coercivity in quotient Hilbert space $\mathcal{H}(\mathbb{R}^N)$

For all $[u] \in \mathcal{H}(\mathbb{R}^N)$, denote $u \in \mathcal{H}^w(\mathbb{R}^N)$ any element in the class $[u]$. Endow $\mathcal{H}(\mathbb{R}^N)$ with the semi-norm

$$|[u]|_{\mathcal{H}(\mathbb{R}^N)} := \|\nabla u\|_{L^2(\mathbb{R}^N)}.$$

Corollary 3.6.1. *The semi-norm $|\ .\ |_{\mathcal{H}(\mathbb{R}^N)}$ and the norm $\|\ .\ \|_{\mathcal{H}(\mathbb{R}^N)}$ are equivalent in $\mathcal{H}(\mathbb{R}^N)$. Moreover*

$$\lim_{\|[u]\|_{\mathcal{H}(\mathbb{R}^N)}\to+\infty} \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{\|[u]\|_{\mathcal{H}(\mathbb{R}^N)}} \;=\; +\infty.$$

Proof. Regarding the equivalence of the norm and of the semi-norm in $\mathcal{H}(\mathbb{R}^N)$, it suffices to apply Corollary 3.5.1 in the particular case $p = 2$.

Due to this equivalence, the claimed limit becomes obvious in the case $p = 2$. It is not even necessary to apply Proposition 3.5.2 to prove it. □

Chapter 4

Topological asymptotic expansion for a class of quasilinear elliptic equations

In this chapter, we first define in section 4.1 a class of non-quadratic potentials such that the variation of the gradient field satisfies a combined p- and 2-ellipticity property as expected according to the conclusions of chapter 2. Then section 4.2 describes the perturbed quasilinear elliptic equations we consider.

Our main contribution is to provide the topological asymptotic expansion stated in Theorem 4.3.1 on page 56.

So as to prove Theorem 4.3.1, we study the variation of the direct state in section 4.4 and the variation of the adjoint state in section 4.5. Along the way, we shall have to:

1. ensure duality between the variation of the direct state and the corresponding variation of the adjoint state, at each stage of approximation;
2. determine the asymptotic behavior of the variation of the direct state at scale 1;
3. determine with respect to the variation of the direct state, what does mean 'far away from the perturbation' by opposition to 'close to the perturbation'.

Then we study the asymptotic expansion of the functional in section 4.6, separating a classical linear term and a new nonlinear term. An operator of nonlinearity, denoted S hereafter, plays a key role in the definition of the nonlinear term of the asymptotic expansion.

For reader's convenience, proofs requiring longer calculations are postponed to section 4.7.

Eventually, with respect to topological asymptotic expansions for quasilinear elliptic equations of second order, conclusions of our research at this stage will be found in section 4.8.

In all this chapter, let $N \in \mathbb{N}$, $N \geq 2$. Let $p \in [2, \infty)$ and q such that $1/p + 1/q = 1$.

4.1 Class of non-quadratic potentials

Let $W : \mathbb{R}^N \to \mathbb{R}$ be a twice Fréchet differentiable potential. Denote $T : \mathbb{R}^N \to \mathbb{R}^N$ the vector field that derives from W, that is:

$$T(\varphi).\psi := DW(\varphi)\psi, \quad \forall\, \varphi, \psi \in \mathbb{R}^N.$$

At the next order of derivation, for all $\varphi \in \mathbb{R}^N$, we define $S_\varphi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$S_\varphi(\psi) := T(\varphi + \psi) - T(\varphi) - DT(\varphi).\psi, \quad \forall\, \psi \in \mathbb{R}^N.$$

Due to the arguments expounded in chapter 2, we choose a class of potentials which in particular ensures both p- and 2-coercivities to the variational form defining the variation of the direct state at scale 1. More precisely, in all this chapter 4, we make the following assumption.

Assumption 4.1.1. The potential W satisfies the following conditions:

1. $W \in C^{2,\alpha}(\mathbb{R}^N, \mathbb{R})$, for some $\alpha > 0$.
2. There exist $b_0 > a_0 > 0$ such that
$$a_0 \,|\varphi|^p \le W(\varphi) \le b_0(1 + |\varphi|^p), \quad \forall\, \varphi \in \mathbb{R}^N.$$
3. There exists $a_1 > 0$ such that
$$|T(\varphi)| \le a_1 \,|\varphi|\,(1 + |\varphi|^{p-2}), \quad \forall\, \varphi \in \mathbb{R}^N.$$
4. There exist $0 < c < C$ such that
$$c(1 + |\varphi|^2)^{\frac{p-2}{2}}\,|\psi|^2 \le DT(\varphi)\psi.\psi \le C(1 + |\varphi|^2)^{\frac{p-2}{2}}\,|\psi|^2\,, \quad \forall\, \varphi, \psi \in \mathbb{R}^N.$$
5. There exists $c > 0$ such that
$$\left(T(\varphi + \psi) - T(\varphi)\right).\psi \ge c(|\psi|^p + |\psi|^2), \quad \forall\, \varphi, \psi \in \mathbb{R}^N.$$
6. There exists $C > 0$ such that
$$|T(\varphi + \psi) - T(\varphi)| \le C\,|\psi|\left[1 + |\varphi|^{p-2} + |\psi|^{p-2}\right], \quad \forall\, \varphi, \psi \in \mathbb{R}^N.$$
7. Let $M > 0$. Then there exist $c_0 = c_0(M, p) \ge 0$ and $c_{p-3} = c_{p-3}(p) \ge 0$ such that
$$|S_\varphi(\psi_2) - S_\varphi(\psi_1)| \le |\psi_2 - \psi_1|\,(|\psi_1| + |\psi_2|)\left[c_0 + c_{p-3}\,(|\psi_1| + |\psi_2|)^{p-3}\right], \quad \forall \varphi \in B(0, M),\ \forall \psi_1, \psi_2 \in \mathbb{R}^N.$$

   In addition for all $M > 0$, the constants c_0 and c_{p-3} satisfy the conditions
$$\begin{cases} c_{p-3} = 0, & \forall p \in [2, 3]\,, \\ c_0 = 0, & \text{if } p = 2. \end{cases}$$

8. Let $M > 0$. Then there exist $d_0 = d_0(M,p) \geq 0$ and $d_{p-4} = d_{p-4}(p) \geq 0$ such that

$$|S_{\varphi_2}(\psi) - S_{\varphi_1}(\psi)| \leq |\varphi_2 - \varphi_1| \, |\psi|^2 \left[d_0 + d_{p-4} \, |\psi|^{p-4}\right], \quad \forall \varphi_1, \varphi_2 \in B(0,M), \; \forall \psi \in \mathbb{R}^N.$$

In addition for all $M > 0$, the constants d_0 and d_{p-4} satisfy the conditions

$$\begin{cases} d_{p-4} = 0, & \forall p \in [2,4]\,, \\ d_0 = 0, & \text{if } p = 2. \end{cases}$$

Let us comment the conditions stated in Assumption 4.1.1.

1. The three conditions (2), (3) and (4) of Assumption 4.1.1 are classical growth conditions about respectively the potential W, the gradient field T and the hessian DT. Such conditions are common in works related to solutions of nonlinear elliptic equations (e.g. [46, 62]). Condition (4) entails that potential W is strictly convex, it also provides 2-ellipticity to variational problems defining the adjoint state and its variations.
2. As expected, condition (5) ensures the combined p- and 2-ellipticity properties applied to define the variation of the direct state at scale 1 in $\mathbb{R}^N$.
3. Condition (6) will be essential to estimate the variations of the direct state at various steps of approximation. It provides much more than a control over the hessian of W, as the upper bound holds for all $\psi \in \mathbb{R}^N$. In particular, when parameter φ is bounded, according to condition (6) it holds:
let $M > 0$, then there exist $b_1 > 0$ and $b_{p-1} > 0$ such that

$$|T(\varphi + \psi) - T(\varphi)| \leq b_1 \, |\psi| + b_{p-1} \, |\psi|^{p-1}\,, \quad \forall\, \varphi \in B(0,M), \forall\, \psi \in \mathbb{R}^N. \tag{4.1.1}$$

Note that, should we have made the additional assumption that $T(0) = 0$, then condition (6) would have implied condition (3). Therefore one could equivalently assume $T(0) = 0$ and condition (6) and forget about assuming condition (3).
4. Conditions (7) and (8) are very much specific of the nonlinearity of the considered problem. Indeed the map S_φ accounts for the nonlinearity of gradient field T at point φ.
    (a) Condition (7) gives control over nonlinearity of gradient field T at a given point φ and involves the third derivative of W, if it exists. In particular, it provides a growth condition about S as follows:
    let $M > 0$, then there exist two constants $c_0 \geq 0$ and $c_{p-3} \geq 0$ satisfying nullity conditions stated in condition (7), such that

    $$|S_\varphi(\psi)| \leq c_0 \, |\psi|^2 + c_{p-3} \, |\psi|^{p-1} \quad \forall \varphi \in B(0,M), \forall \psi \in \mathbb{R}^N. \tag{4.1.2}$$

    (b) Condition (8) takes into account the fourth derivative of W, if it exists, and accounts for the variation of the nonlinearity of gradient field T from a given point φ_1 to another point φ_2.

5. In both conditions (7) and (8), the cases of nullity stated for constants c_0, c_{p-3}, d_0, d_{p-4} follow from the fact that for smaller values of p, some terms appearing after differentiation can be bounded from above.

   Obviously choosing $p = 2$ brings back to the linear case with $S_\varphi = 0$, for all $\varphi \in \mathbb{R}^N$.

In the subsequent we shall only write 'condition (i)' instead of 'condition (i) of Assumption 4.1.1' whenever no confusion is possible.

The class of potentials satisfying Assumption 4.1.1 encompasses the archetype of non-degenerate elliptic potentials ([46] p.261 or [91] p.343) given by

$$W_a \quad : \quad \varphi \in \mathbb{R}^N \mapsto \frac{1}{p}\left(a^2 + |\varphi|^2\right)^{p/2}, \tag{4.1.3}$$

where $a > 0$.

In the case of potential W_a, the p- and 2-ellipticity properties required by condition (5) follow from a slightly extended version of an inequality given by Lindqvist in [60], page 71 (I).

Proposition 4.1.2. *Let $a > 0$ and $p \in [2, \infty)$. Then there exists $c > 0$ such that*

$$\left[(a^2 + |\varphi + \psi|^2)^{\frac{p-2}{2}}(\varphi + \psi) - (a^2 + |\varphi|^2)^{\frac{p-2}{2}}\varphi\right].\psi \geq c\,(|\psi|^p + |\psi|^2), \quad \forall\, \varphi, \psi \in \mathbb{R}^N.$$

The proof is available in subsection 4.7.1 on page 70. It is easy to check that the p-ellipticity does not depend on a while the 2-ellipticity vanishes when $a \to 0$.

At the price of some calculations, it follows

Proposition 4.1.3. *Let $a > 0$ and $p \in [2, \infty)$. Then potential W_a satisfies Assumption 4.1.1.*

The proof is available in subsection 4.7.1 on page 71.

4.2 The perturbed nonlinear equation

Let Ω be a bounded domain of $\mathbb{R}^N$, with a C^2-boundary $\partial\Omega$.

Let a function $f \in C^{0,\alpha}(\Omega)$, for some $\alpha > 0$ and with a support $spt(f) \subset\subset \Omega$. Let a point $x_0 \in \Omega \setminus spt(f)$. For simplicity and without loss of generality, we assume that $x_0 = 0$.

Consider a bounded domain ω of $\mathbb{R}^N$ with a C^2-boundary $\partial\omega$ and such that $0 \in \omega$. For all $\varepsilon \geq 0$, let $\omega_\varepsilon := x_0 + \varepsilon\omega$. In all this chapter, we assume $\varepsilon \geq 0$ is small enough such that $\omega_\varepsilon \subset\subset \Omega \setminus spt(f)$. Moreover changing if appropriate ω (resp. ε) into ω/λ (resp. into $\lambda\varepsilon$) for some $\lambda > 0$ large enough, we can assume without loss of generality that there exist two real numbers

$$0 < \rho < R \quad \text{such that} \quad \omega \subset\subset B(0,\rho) \subset B(0,R) \subset\subset \Omega \setminus spt(f). \tag{4.2.1}$$

Let the conductivity function $\gamma_\varepsilon : \Omega \to \mathbb{R}$ given by

$$\gamma_\varepsilon := \gamma_0 \text{ in } \Omega \setminus \omega_\varepsilon \quad \text{and} \quad \gamma_\varepsilon := \gamma_1 \text{ in } \omega_\varepsilon,$$

where γ_0, γ_1 are two positive real numbers with $\gamma_0 \neq \gamma_1$. Denote $\underline{\gamma} := \min(\gamma_0, \gamma_1)(> 0)$ and $\overline{\gamma} := \max(\gamma_0, \gamma_1)$.

We define the direct state in the space $\mathcal{V} := W_0^{1,p}(\Omega)$.

Lemma 4.2.1. *For all $\varepsilon \geq 0$ small enough, the functional*

$$\mathcal{W}_\varepsilon : \eta \in \mathcal{V} \mapsto \int_\Omega \gamma_\varepsilon W(\nabla \eta) \; - \; \int_\Omega f\eta$$

is Fréchet differentiable, strictly convex and coercive in $\mathcal{V}$. Thus we define u_ε as

$$\{u_\varepsilon\} = \operatorname*{argmin}_{\eta \in \mathcal{V}} \mathcal{W}_\varepsilon(\eta).$$

This solution is characterized by the Euler-Lagrange equation:

$$\text{find } u_\varepsilon \in \mathcal{V} \text{ such that } \int_\Omega \gamma_\varepsilon T(\nabla u_\varepsilon).\nabla \eta = \int_\Omega f\eta, \quad \forall \eta \in \mathcal{V}, \tag{4.2.2}$$

which strong form is

$$\text{find } u_\varepsilon \in W^{1,p}(\Omega) \text{ such that} \begin{cases} -\operatorname{div}\left(\gamma_\varepsilon T(\nabla u_\varepsilon)\right) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof is standard and is made available in subsection 4.7.2 on page 75.

4.3 Topological asymptotic expansion

For $\varepsilon \geq 0$ small enough, let $\mathcal{J}_\varepsilon : \mathcal{H} \to \mathbb{R}$ a functional such that

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_0(u_0) + \langle G, u_\varepsilon - u_0 \rangle + \delta_2 \, \varepsilon^N + R(\varepsilon), \tag{4.3.1}$$

where

1. G denotes a bounded linear form on $\mathcal{H}$;
2. $\delta_2 \in \mathbb{R}$;
3. the remainder $R(\varepsilon)$ is
   (a) either of the form

   $$R(\varepsilon) = o\left(\|u_\varepsilon - u_0\|^2_{\mathcal{H}}\right), \tag{4.3.2}$$

   (b) or of the form

   $$R(\varepsilon) = O\left(\int_{\Omega \setminus B(0,\tilde{\alpha}\varepsilon^{\tilde{r}})} |\nabla(u_\varepsilon - u_0)|^p + |\nabla(u_\varepsilon - u_0)|^2\right), \tag{4.3.3}$$

   for a given $\tilde{\alpha} > 0$ and a given $\tilde{r} \in (0,1)$. In this second case, the remainder $R(\varepsilon)$ is bounded by the variation of the direct state 'far away' from the perturbation ω_ε.

A classical example of such a functional is given by the compliance

$$u \mapsto \int_{\operatorname{spt}(f)} fu, \quad \forall\, u \in \mathcal{H},$$

with in this particular case, $G = f$ and $\delta_2 = 0$.

We now have all the ingredients to state our main result which provides the topological asymptotic expansion of $\mathcal{J}_\varepsilon(u_\varepsilon)$. We denote:

- the function u_0 is the unperturbed direct state defined by (4.2.2) in the case $\varepsilon = 0$;
- the vector $U_0 := \nabla u_0(x_0)$ is the gradient of u_0 at point x_0;
- the function H is the variation of the direct state at scale 1 in $\mathbb{R}^N$ defined by (4.4.5);
- the function v_0 is the unperturbed adjoint state defined by (4.5.1) in the case $\varepsilon = 0$;
- the vector $V_0 := \nabla v_0(x_0)$ is the gradient of v_0 at point x_0;
- the function K is the variation of the adjoint state at scale 1 in $\mathbb{R}^N$ defined by (4.5.6);
- the conductivity function γ at scale 1 is defined by

$$\gamma := \gamma_0 \text{ in } \mathbb{R}^N \setminus \omega \quad \text{and} \quad \gamma := \gamma_1 \text{ in } \omega; \tag{4.3.4}$$

- $\mathcal{P}$ is the polarization tensor defined by (4.6.10) and which only depends on the set ω, on the definite positive matrix $DT(U_0)$ and on the ratio γ_1/γ_0.

Theorem 4.3.1. *Assume that*
- *the potential W satisfies Assumption 4.1.1;*
- *the functional $(\mathcal{J}_\varepsilon)$ satisfies an expansion of the type (4.3.1);*
- *the direct unperturbed state satisfies $u_0 \in L^\infty(\Omega)$;*
- *the unperturbed adjoint state satisfies $v_0 \in L^\infty(\Omega)$, $\nabla v_0 \in L^\infty(\Omega)$ and v_0 and ∇v_0 Hölder continuous at point x_0;*
- *the variation of the direct state at scale 1 satisfies the asymptotic behavior stated in Assumption 4.4.10 on page 63.*

Then for $\varepsilon > 0$ small enough it holds

$$\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) = \varepsilon^N g(x_0) + o(\varepsilon^N), \tag{4.3.5}$$

with

$$\begin{aligned} g(x_0) \;&:=\; T(U_0)^T \mathcal{P} V_0 + \delta_2 && (4.3.6)\\ &\quad + \int_{\mathbb{R}^N} \gamma S_{U_0}(\nabla H).\,(V_0 + \nabla K)\,. && (4.3.7) \end{aligned}$$

Two terms (4.3.6) and (4.3.7) appear in the formula on the topological gradient.
- In the linear case, where $S_{U_0} = 0$, the topological gradient $g(x_0)$ reduces to the first term 4.3.6. It can thus be estimated by computing the gradient field of the unperturbed direct state u_0, the gradient field of the unperturbed adjoint state v_0 and the polarization tensor $\mathcal{P}$, which however also depends on $DT(U_0)$.
- The term (4.3.7) is published here for the first time. It accounts for the component of the topological gradient which is caused by the nonlinearity of the equation. With respect to applications, an important issue will be the cost of its computation.

It is worth emphasizing that the regularity assumptions made in Theorem 4.3.1 are much weaker than what they may seem at first glance, as:
- In the main case $\omega = B(0,1)$, $W = W_a$ for some $a > 0$ and $\gamma_1 < \gamma_0$, let

$$\bar{p} := 2 + \left(1 + \frac{a^2}{|U_0|^2}\right) \frac{N}{N-2},$$

with the convention that $\bar{p} = +\infty$ when $N = 2$.
If $p \in [2, \bar{p})$, then no assumption has to be made about the asymptotic behavior of function H, as it is then ensured by virtue of Theorem 4.4.9 on page 62.

- The assumption $u_0 \in L^\infty(\Omega)$ is theoretically needed for proving the $C^{1,\beta}(\overline{\Omega})$ regularity of u_0 (see Lemma 4.4.1 on page 57). But from a practical point of view, this assumption could be taken for granted.
- When G is regular enough, Lemma 4.5.1 on page 64 states that v_0 in $C^{1,\tilde{\beta}}(\overline{\Omega})$. In such a case, it is not necessary to make any assumption about the regularity of v_0 in Theorem 4.3.1.

So as to prove Theorem (4.3.1), we now have to study

1. the variation of the direct state in section 4.4 hereafter;
2. the variation of the adjoint state in section 4.5 on page 64;
3. the asymptotic expansion of $\mathcal{J}_\varepsilon(u_\varepsilon)$ in section 4.6 on page 66.

4.4 Variation of the direct state

4.4.1 About the regularity of the unperturbed direct state

In the unperturbed case $\varepsilon = 0$, Euler-Lagrange equation (4.2.2) reads

$$\int_\Omega \gamma_0 T(\nabla u_0).\nabla\eta = \int_\Omega f\eta, \quad \forall \eta \in \mathcal{V}. \tag{4.4.1}$$

In all the subsequent, we assume that $u_0 \in L^\infty(\Omega)$.

Lemma 4.4.1. *Assume that $u_0 \in L^\infty(\Omega)$. Then it holds $u_0 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta > 0$.*

Proof. The unperturbed direct state $u_0 \in \mathcal{V}$ is weak solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\gamma_0 T(\nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

According to condition (1) it holds $T \in C^{1,\alpha}(\mathbb{R}^N)$ and by assumption $f \in C^{0,\alpha}(\overline{\Omega})$. Moreover referring to [62], structure conditions (3.46) p.181 hold by virtue of condition (4). Hence it follows from [62] Theorem 3.20 that $u_0 \in C^{1,\beta}(\overline{\Omega})$, for some $\beta > 0$. □

4.4.2 Step 1: variation $u_\varepsilon - u_0$

Let $\tilde{u}_\varepsilon := u_\varepsilon - u_0 \in \mathcal{V}$. According to (4.4.1), it is straightforward that one can rewrite Lemma 4.2.1 with respect to $\tilde{u}_\varepsilon$ as follows

Lemma 4.4.2. *For all $\varepsilon \geq 0$ small enough, the functional*

$$\tilde{\mathcal{W}}_\varepsilon : \eta \in \mathcal{V} \mapsto \int_\Omega \gamma_\varepsilon W(\nabla u_0 + \nabla\eta) \; - \; \int_\Omega \gamma_0 T(\nabla u_0).\nabla\eta$$

is Fréchet differentiable, strictly convex and coercive in $\mathcal{V}$. The following equality holds

$$\{\tilde{u}_\varepsilon\} = \operatorname*{argmin}_{\eta \in \mathcal{V}} \tilde{\mathcal{W}}_\varepsilon(\eta).$$

Thus $\tilde{u}_\varepsilon$ is characterized by the Euler-Lagrange equation: find $\tilde{u} \in \mathcal{V}$ such that

$$\int_\Omega \gamma_\varepsilon T(\nabla u_0 + \nabla \tilde{u}).\nabla \eta = \int_\Omega \gamma_0 T(\nabla u_0).\nabla \eta, \quad \forall \eta \in \mathcal{V}, \tag{4.4.2}$$

Since $\gamma_\varepsilon - \gamma_0 = \gamma_1 - \gamma_0$ in ω_ε and $\gamma_\varepsilon - \gamma_0 = 0$ in $\Omega \setminus \omega_\varepsilon$, the latter can be rewritten

$$\int_\Omega \gamma_\varepsilon \left[T(\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T(\nabla u_0)\right].\nabla \eta + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} T(\nabla u_0).\nabla \eta = 0, \quad \forall \eta \in \mathcal{V}. \tag{4.4.3}$$

4.4.3 Step 2: approximation of variation $u_\varepsilon - u_0$

We approximate $\tilde{u}_\varepsilon$ by h_ε defined as follows. Recall we denote $U_0 = \nabla u_0(0)$.

Lemma 4.4.3. *For all $\varepsilon \geq 0$ small enough, the functional*

$$\mathcal{I}_\varepsilon : \eta \in \mathcal{V} \mapsto \int_\Omega \gamma_\varepsilon W(U_0 + \nabla \eta) \;-\; \int_\Omega \gamma_0 T(U_0).\nabla \eta$$

is Fréchet differentiable, strictly convex and coercive in $\mathcal{V}$. We define

$$\{h_\varepsilon\} = \underset{\eta \in \mathcal{V}}{\operatorname{argmin}}\, \mathcal{I}_\varepsilon(\eta).$$

The solution h_ε is characterized by the Euler-Lagrange equation: find $h \in \mathcal{V}$ such that

$$\int_\Omega \gamma_\varepsilon \left[T(\nabla h + U_0) - T(U_0)\right].\nabla \eta \;+\; (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} T(U_0).\nabla \eta = 0, \quad \forall \eta \in \mathcal{V}. \tag{4.4.4}$$

The proof of Lemma 4.4.3 is similar to that of Lemma 4.2.1.

4.4.4 Step 3: change of scale

Recall the conductivity function $\gamma : \mathbb{R}^N \to \mathbb{R}$ defined by (4.3.4), i.e.

$$\gamma := \gamma_0 \text{ in } \mathbb{R}^N \setminus \omega \quad \text{ and } \quad \gamma := \gamma_1 \text{ in } \omega.$$

For the purpose of defining the variation of the direct state at scale 1 in $\mathbb{R}^N$, we look for a function H which may approximate the map $y \in \Omega/\varepsilon \mapsto \varepsilon^{-1} h_\varepsilon(\varepsilon y)$.

As in the particular case of W_a, it holds $W_a(\psi) \geq \frac{1}{p} a^p > 0$, for all $\psi \in \mathbb{R}^N$, there is no hope to define function H as the minimizer of a functional involving the integral

$$\int_{\mathbb{R}^N} \gamma W(\nabla H).$$

One has to start from variational form (4.4.4). Therefore we apply the Minty-Browder theorem in the reflexive Banach space $\mathcal{V}(\mathbb{R}^N)$. The space $\mathcal{V}(\mathbb{R}^N)$ was studied in section 3.5 on page 49. Corollary 3.5.1 states equivalence between the norm and the semi-norm in $\mathcal{V}(\mathbb{R}^n)$. The main coercivity property in $\mathcal{V}(\mathbb{R}^n)$ is stated in Proposition 3.5.2.

Proposition 4.4.4. *There exists a unique function $H \in \mathcal{V}(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} \gamma \left[T(U_0 + \nabla H) - T(U_0)\right].\nabla \eta = -(\gamma_1 - \gamma_0) \int_\omega T(U_0).\nabla \eta, \quad \forall \eta \in \mathcal{V}(\mathbb{R}^N). \tag{4.4.5}$$

Proof. According to inequality (4.1.1) obtained from condition (6) and after Hölder's inequality, for all $\eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N)$ it holds

$$\int_{\mathbb{R}^N} |\gamma \left[T(U_0 + \nabla\eta_1) - T(U_0)\right] .\nabla\eta_2| \leq \int_{\mathbb{R}^N} \overline{\gamma} \; \left(b_1 |\nabla\eta_1| + b_{p-1} |\nabla\eta_1|^{p-1}\right) |\nabla\eta_2|$$
$$\leq \overline{\gamma} b_1 \|\nabla\eta_1\|_{L^2(\mathbb{R}^N)} \|\nabla\eta_2\|_{L^2(\mathbb{R}^N)} + \overline{\gamma} b_{p-1} \|\nabla\eta_1\|^{\frac{p}{q}}_{L^p(\mathbb{R}^N)} \|\nabla\eta_2\|_{L^p(\mathbb{R}^N)} . \quad (4.4.6)$$

According to Lemma 3.2.2, $\mathcal{V}(\mathbb{R}^N)$ is a reflexive Banach space. Denote $\mathcal{V}^*(\mathbb{R}^N)$ the topological dual of $\mathcal{V}(\mathbb{R}^N)$ and $\langle ., . \rangle$ the duality product between $\mathcal{V}(\mathbb{R}^N)$ and $\mathcal{V}^*(\mathbb{R}^N)$. Then define operator A by

$$\langle A\eta_1, \eta_2 \rangle := \int_{\mathbb{R}^N} \gamma \left[T(U_0 + \nabla\eta_1) - T(U_0)\right] .\nabla\eta_2, \quad \forall\; \eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N). \quad (4.4.7)$$

According to inequality (4.4.6), $\langle A\eta_1, \eta_2 \rangle$ is well defined for all $\eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N)$.

Moreover for all $\eta_1 \in \mathcal{V}(\mathbb{R}^N)$, the map

$$\langle A\eta_1, . \rangle : \eta_2 \in \mathcal{V}(\mathbb{R}^N) \mapsto \langle A\eta_1, \eta_2 \rangle$$

is a bounded linear form with

$$\|A\eta_1\|_{\mathcal{V}^*(\mathbb{R}^N)} \leq \overline{\gamma} \left(b_1 \|\nabla\eta_1\|_{L^2(\mathbb{R}^N)} + b_{p-1} \|\nabla\eta_1\|^{\frac{p}{q}}_{L^p(\mathbb{R}^N)}\right).$$

Hence equation (4.4.7) defines an operator $A : \mathcal{V}(\mathbb{R}^N) \to \mathcal{V}^*(\mathbb{R}^N)$.

Then define $L \in \mathcal{V}^*(\mathbb{R}^N)$ by

$$L : \eta \in \mathcal{V}(\mathbb{R}^N) \mapsto -(\gamma_1 - \gamma_0) \int_{\omega} U_0.\nabla\eta.$$

The variational problem (4.4.5) can be equivalently written: find $H \in \mathcal{V}(\mathbb{R}^N)$ such that $AH = L$. Let us check the assumptions required by the Minty-Browder theorem.

1. Let $\eta_1 \in \mathcal{V}(\mathbb{R}^N)$. According to condition (6) and after Hölder's inequality, for all $\eta, \eta_2 \in \mathcal{V}(\mathbb{R}^N)$ it holds

$$|\langle [A(\eta_1 + \eta) - A\eta_1], \eta_2 \rangle| = \left| \int_{\mathbb{R}^N} \gamma \left[T(U_0 + \nabla(\eta_1 + \eta)) - T(U_0 + \nabla\eta_1)\right] .\nabla\eta_2 \right|$$
$$\leq \overline{C} \int_{\mathbb{R}^N} |\nabla\eta| \left[1 + |U_0 + \nabla\eta_1|^{p-2} + |\nabla\eta|^{p-2}\right] |\nabla\eta_2|$$
$$\leq \overline{C} \int_{\mathbb{R}^N} |\nabla\eta| \left[\left(1 + 2^{p-2} |U_0|^{p-2}\right) + 2^{p-2} |\nabla\eta_1|^{p-2} + |\nabla\eta|^{p-2}\right] |\nabla\eta_2|$$
$$\leq \overline{C} \left(1 + 2^{p-2} |U_0|^{p-2}\right) \|\nabla\eta\|_{L^2(\mathbb{R}^N)} \|\nabla\eta_2\|_{L^2(\mathbb{R}^N)}$$
$$+ \; \overline{C} \left[2^{p-2} \|\nabla\eta_1\|^{p-2}_{L^p(\mathbb{R}^N)} \|\nabla\eta\|_{L^p(\mathbb{R}^N)} + \|\nabla\eta\|^{\frac{p}{q}}_{L^p(\mathbb{R}^N)}\right] \|\nabla\eta_2\|_{L^p(\mathbb{R}^N)},$$

where $\overline{C} := C\; \overline{\gamma}$ and C is the constant of condition (6).

Hence

$$\|A(\eta_1 + \eta) - A\eta_1\|_{\mathcal{V}^*(\mathbb{R}^N)} \leq \tilde{C} \left[\|\nabla\eta\|_{L^2(\mathbb{R}^N)} + \|\nabla\eta_1\|^{p-2}_{L^p(\mathbb{R}^N)} \|\nabla\eta\|_{L^p(\mathbb{R}^N)} + \|\nabla\eta\|^{\frac{p}{q}}_{L^p(\mathbb{R}^N)}\right],$$

where $\tilde{C} := \overline{C} \max\left(1 + 2^{p-2} |U_0|^{p-2}, 2^{p-2}\right)$.

It follows that $A(\eta_1 + \eta) - A\eta_1 \to 0$ in $\mathcal{V}^*(\mathbb{R}^N)$ when $\eta \to 0$ in $\mathcal{V}(\mathbb{R}^N)$. Thus A is continuous at point η_1, for all $\eta_1 \in \mathcal{V}(\mathbb{R}^N)$. It follows that A is continuous.

2. According to condition (5), there exists $c > 0$ such that for all $\eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N)$,
$$\begin{aligned} \langle A\eta_1 - A\eta_2, \eta_1 - \eta_2 \rangle &= \int_{\mathbb{R}^N} \gamma \left[T(U_0 + \nabla\eta_1) - T(U_0 + \nabla\eta_2) \right] .(\nabla\eta_1 - \nabla\eta_2) \\ &\geq c\underline{\gamma} \left(\|\nabla\eta_1 - \nabla\eta_2\|^p_{L^p(\mathbb{R}^N)} + \|\nabla\eta_1 - \nabla\eta_2\|^2_{L^2(\mathbb{R}^N)} \right). \end{aligned}$$
Hence
$$\langle A\eta_1 - A\eta_2, \eta_1 - \eta_2 \rangle \geq 0, \quad \forall \eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N).$$
In addition
$$\langle A\eta_1 - A\eta_2, \eta_1 - \eta_2 \rangle = 0$$
implies that $\nabla\eta_1 = \nabla\eta_2$ a.e. in $\mathbb{R}^N$ and thus $\eta_1 = \eta_2$ in the quotient space $\mathcal{V}(\mathbb{R}^N)$. Hence A is strictly monotone.
3. Lastly according to condition (5), there exists $c > 0$ such that for all $\eta \in \mathcal{V}(\mathbb{R}^N)$ it holds
$$\begin{aligned} \langle A\eta, \eta \rangle &= \int_{\mathbb{R}^N} \gamma \left[T(U_0 + \nabla\eta) - T(U_0) \right] .\nabla\eta \\ &\geq c\underline{\gamma}(\|\nabla\eta\|^p_{L^p(\mathbb{R}^N)} + \|\nabla\eta\|^2_{L^2(\mathbb{R}^N)}). \end{aligned}$$
Thus it follows from Proposition 3.5.2 that
$$\lim_{\|\eta\|\to\infty} \frac{\langle A\eta, \eta \rangle}{\|\eta\|_{\mathcal{V}(\mathbb{R}^N)}} = +\infty.$$
Hence A is coercive in $\mathcal{V}(\mathbb{R}^N)$.

Therefore according to the Minty-Browder theorem ([30], Theorem V-15), there exists a unique $H \in \mathcal{V}(\mathbb{R}^N)$ such that $AH = L$, which completes the proof of Proposition 4.4.4. □

4.4.5 Step 4: asymptotic behavior of variations of the direct state

For all $\varepsilon > 0$ small enough, let
$$H_\varepsilon : x \in \Omega \mapsto H_\varepsilon(x) := \varepsilon \hat{H}(\varepsilon^{-1}x) \tag{4.4.8}$$
where $\hat{H} \in \mathcal{V}^w(\mathbb{R}^N)$ denotes a given element of the class $H \in \mathcal{V}(\mathbb{R}^N)$.

Making the change of scale backward, as
$$\inf_{x\in\Omega} w_p \left(\frac{x}{\varepsilon} \right) > 0,$$
it follows from $\hat{H} \in \mathcal{V}^w(\mathbb{R}^N)$ that $H_\varepsilon \in W^{1,p}(\Omega) \subset H^1(\Omega)$.

After the combined p- and 2-ellipticity properties stated in condition (5), it holds

Lemma 4.4.5.
$$\|\nabla\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N), \tag{4.4.9}$$
$$\|\nabla h_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N), \tag{4.4.10}$$
$$\|\nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla H_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N). \tag{4.4.11}$$

The proof is available in subsection 4.7.2 on page 76.

Further estimation of the variation of the direct state at scale ε requires to estimate the asymptotic behavior of function H at scale 1 in $\mathbb{R}^N$. To our best knowledge, no such result is available in the literature, e.g. [71, 72, 84]. In [76], Rabier and Stuart proved an exponential decay of solutions, but for a class of quasilinear equations which does not encompass the case of function H.

Let us first study the asymptotic decay of a radial function of $\mathcal{V}(\mathbb{R}^N)$. Let $\eta \in \mathcal{V}^w(\mathbb{R}^N)$ such that for some $C, \tau \in \mathbb{R}$ and $M > 0$, it holds

$$\eta(x) = C\,|x|^{-\tau}\,, \quad \forall x \in \mathbb{R}^N, |x| \geq M.$$

By definition of $\mathcal{V}^w(\mathbb{R}^N)$, it holds $w_p\eta \in L^p(\mathbb{R}^N)$ and $\nabla\eta \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. An easy calculation shows that these conditions imply

$$\tau > \frac{N}{2} - 1$$

and that, whatever the values of $N \geq 2$ and $p \in [2, \infty)$, the constraint $\nabla\eta \in L^2(\mathbb{R}^N)$ is always the active one with respect to exponent τ.

In the following subsubsection, we are going to prove that the asymptotic decay of function H is similar to that of a radial function of $\mathcal{V}(\mathbb{R}^N)$, at least in the main case, that is when the perturbation subdomain ω is the unit ball $B(0,1)$ and when the potential W is a prototype potential W_a as defined by (4.1.3). Accordingly, relaxing the assumption about the shape of ω and assuming only that potential W belongs to the class of potentials defined by Assumption 4.1.1, we shall make the Assumption 4.4.10 hereafter about the asymptotic behavior of function H.

Asymptotic behavior of function H in the case $\omega = B(0,1)$ and $W = W_a$.

Denote Q the quasilinear elliptic operator such that $QH = 0$, that is

$$\langle Q\eta_1, \eta_2\rangle := \int_{\mathbb{R}^N} \gamma\left[T(U_0 + \nabla\eta_1) - T(U_0)\right].\nabla\eta_2 + (\gamma_1 - \gamma_0)\int_\omega T(U_0).\nabla\eta_2,$$
$$\forall \eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N). \quad (4.4.12)$$

We assume $\omega = B(0,1)$ and $W = W_a$ for some $a > 0$. Assuming again the non trivial case $U_0 \neq 0$, let $e_1 = |U_0|^{-1} U_0$. Let $(e_1, e_2, ..., e_N)$ an orthonormal basis of $\mathbb{R}^N$. Denote $(x_1, x_2, ..., x_N)$ the system of coordinates in this basis.

Denote $\mathbb{R}^N_+$ the half-space $\left\{x \in \mathbb{R}^N; U_0.\ x \geq 0\right\}$. Due to the symmetry of $\omega = B(0,1)$ with respect to the line $\mathbb{R}U_0$, it follows straightforwardly from the uniqueness stated in Proposition 4.4.4, that H is odd with respect to the first coordinate. In other words, there exists an element $\tilde{H}$ of the class H such that

$$\tilde{H}(-x_1, x_2, ..., x_N) = -\tilde{H}(x_1, x_2, ..., x_N), \quad \forall (x_1, x_2, ..., x_N) \in \mathbb{R}^N.$$

In particular

$$\tilde{H}(0, x_2, ..., x_N) = 0, \quad \forall (x_2, ..., x_N) \in \mathbb{R}^{N-1}.$$

Hence it suffices to study the asymptotic behavior of function $\tilde{H}$ in the half-space $\mathbb{R}^N_+$.

We denote

$$\bar{p} := 2 + \left(1 + \frac{a^2}{|U_0|^2}\right) \frac{N}{N-2}, \tag{4.4.13}$$

with the convention that $\bar{p} = +\infty$ if $N = 2$. The following proposition states that, if $p \in [2, \bar{p})$, then there exists a supersolution P of quasilinear operator Q in the half-space $\mathbb{R}^N_+$, such that $P = O(|x|^{-\tau})$ at infinity, for some $\tau > N/2 - 1$.

Proposition 4.4.6. *Assume $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ and $W = W_a$ for some $a > 0$. If $p \in [2, \bar{p})$, then there exists $\beta > N/2$ and a function $P \in \mathcal{V}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that*

$$\begin{aligned} P(x) &:= k(U_0.x)\,|x|^{-\beta}, \qquad \forall x \in \mathbb{R}^N, |x| > 1; \\ P(x) &:= k(U_0.x), \qquad \forall x \in \mathbb{R}^N, |x| \leq 1; \\ \langle QP, \eta \rangle &\geq 0, \qquad \forall \eta \in \mathcal{V}(\mathbb{R}^N), \textit{support}(\eta) \subset \mathbb{R}^N_+, \eta \geq 0 \ \textit{a.e.}\ , \end{aligned}$$

where

$$k := \frac{\gamma_0 - \gamma_1}{\gamma_1 + \gamma_0(\beta - 1)}. \tag{4.4.14}$$

The proof is made available in subsection 4.7.2 on page 77.

The following lemma states that similarly the null function is a subsolution of operator Q in the half space $\mathbb{R}^N_+$.

Lemma 4.4.7. *Assume $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ and $W = W_a$ for some $a > 0$. Denote* 0 *the null function in $\mathbb{R}^N$. Then*

$$\langle Q0, \eta \rangle \leq 0, \qquad \forall \eta \in \mathcal{V}(\mathbb{R}^N), \textit{support}(\eta) \subset \mathbb{R}^N_+, \eta \geq 0 \ \textit{a.e.}\ .$$

The proof is straightforward following the steps taken in the proof of Proposition (4.4.6). Only is the transmission condition to be checked across $\partial\omega$, as obviously $Q0 = 0$ in ω and in $\mathbb{R}^N \setminus \bar{\omega}$.

Proposition 4.4.8. *Assume $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ and $W = W_a$ for some $a > 0$ and $p \in [2, \bar{p})$. Let P the supersolution defined in Proposition (4.4.6). Then there exists an element $\tilde{H}$ of the class H such that*

$$0 \leq \tilde{H}(x) \leq P(x), \quad \textit{for a.e. } x \in \mathbb{R}^N_+. \tag{4.4.15}$$

The proof is available in subsection 4.7.2 on page 82. Therefore we can now state

Theorem 4.4.9. *Assume $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ and $W = W_a$ for some $a > 0$. Assume $p \in [2, \bar{p})$ where $\bar{p}$ is defined by equation (4.4.13).*
Then there exists an element $\tilde{H}$ of the class $H \in \mathcal{V}(\mathbb{R}^N)$ and $\tau > \frac{N}{2} - 1$ such that

$$\tilde{H}(y) = O\left(|y|^{-\tau}\right) \quad \textit{as} \quad |y| \to +\infty. \tag{4.4.16}$$

Moreover

$$H \in L^\infty(\mathbb{R}^N). \tag{4.4.17}$$

This completes the subsection about the asymptotic behavior of H in the case $\omega = B(0,1)$ and $W = W_a$. Accordingly we make the following assumption in the general case:

Assumption 4.4.10. We assume that

1. there exists an element $\tilde{H}$ of the class $H \in \mathcal{V}(\mathbb{R}^N)$ and $\tau > \frac{N}{2} - 1$ such that

$$\tilde{H}(y) = O\left(|y|^{-\tau}\right) \quad \text{as} \quad |y| \to +\infty; \tag{4.4.18}$$

2. and

$$H \in L^\infty(\mathbb{R}^N). \tag{4.4.19}$$

Lemma 4.4.11. *It holds $H \in \mathcal{H}(\mathbb{R}^N)$*

The proof is available in subsection 4.7.2 on page 83.

From now on, function H_ε is defined choosing $\hat{H} = \tilde{H}$ in (4.4.8), i.e.

$$H_\varepsilon(x) := \varepsilon \tilde{H}(\varepsilon^{-1}x), \quad \forall x \in \Omega. \tag{4.4.20}$$

Proposition 4.4.12. *It holds*

$$\begin{aligned}
\|\nabla h_\varepsilon - \nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^2_{L^2(\Omega)} &= o(\varepsilon^N), && (4.4.21)\\
\forall \alpha > 0,\ \forall r \in (0,1),\ \int_{\Omega \setminus B(0,\alpha\varepsilon^r)} |\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2 &= o(\varepsilon^N), && (4.4.22)\\
\int_\Omega |\nabla u_0 - U_0| \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) &= o(\varepsilon^N), && (4.4.23)\\
\forall\ p \in (4,\infty), \int_\Omega |\nabla u_0 - U_0|\, |\nabla h_\varepsilon|^{p-2} &= o(\varepsilon^N), && (4.4.24)\\
\forall\ p \in (3,\infty), \int_\Omega |\nabla u_0 - U_0|\, |\nabla h_\varepsilon|^{p-1} &= o(\varepsilon^N), && (4.4.25)\\
\int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right) &= o(\varepsilon^N), && (4.4.26)\\
\forall\ p \in (3,\infty), \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right)^{p-2} &= o(\varepsilon^N). && (4.4.27)
\end{aligned}$$

The proof is available in subsection 4.7.2 on page 83.

Proposition 4.4.13. *It holds*

$$\begin{aligned}
\|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|^2_{L^2(\Omega)} &= o(e^N), && (4.4.28)\\
\int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right) &= o(\varepsilon^N), && (4.4.29)\\
\forall\ p \in (3,\infty), \int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right)^{p-2} &= o(\varepsilon^N), && (4.4.30)\\
\forall \alpha > 0,\ \forall\ r \in (0,1),\ \int_{\Omega \setminus B(0,\alpha\varepsilon^r)} |\nabla \tilde{u}_\varepsilon|^p + |\nabla \tilde{u}_\varepsilon|^2 &= o(\varepsilon^N). && (4.4.31)
\end{aligned}$$

The proof is available in subsection 4.7.2 on page 87. Estimates stated in Propositions 4.4.12 and 4.4.13 will be applied to obtain the topological asymptotic expansion in subsection 4.7.4. Estimates (4.4.22) and (4.4.31) state that the energy of the variation outside a ball of radius $\alpha\varepsilon^r$, $r \in (0,1)$, can be neglected at first order in the asymptotic expansion. When $\varepsilon \to 0$, the radius of the ball $B(0, \alpha\varepsilon^r)$ goes to 0 but at the scale of the perturbation sub-domain ω_ε, its boundary goes to infinity. The radius $\alpha\varepsilon^r$ follows directly from the asymptotic behavior of function H.

4.5 Variation of the adjoint state

The adjoint state is defined in the Hilbert space $\mathcal{H} = H_0^1(\Omega)$. According to coercivity provided by condition (4), one may apply the Lax-Milgram theorem in $\mathcal{H}$ and obtains that, for all $\varepsilon \geq 0$ small enough, there exists a unique $v_\varepsilon \in \mathcal{H}$ solution of variational problem

$$\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla v_\varepsilon.\nabla\eta = -\langle G, \eta\rangle, \quad \forall \eta \in \mathcal{H}. \tag{4.5.1}$$

4.5.1 About the regularity of the unperturbed adjoint state

Lemma 4.5.1. *If the functional $\mathcal{J}_\varepsilon$ is the compliance, then it holds v_0 in $C^{1,\tilde\beta}(\overline{\Omega})$ for some $\tilde\beta > 0$. In particular $v_0 \in L^\infty(\Omega)$, $\nabla v_0 \in L^\infty(\Omega)$ and $v_0 \in \mathcal{V}$.*

Proof. As $\mathcal{J}_\varepsilon$ is assumed to be the compliance, it holds $G = f \in C^{0,\alpha}(\Omega)$. The unperturbed adjoint state $v_0 \in \mathcal{H}$ is weak solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\gamma_0 DT(\nabla u_0)\nabla v_0)) = -f & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5.2}$$

We refer to [46], chap. 8. According to condition (1), it holds $DT \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R})$ and after Lemma 4.4.1, it holds $\nabla u_0 \in C^{0,\beta}(\overline{\Omega})$. Hence $DT(\nabla u_0) \in C^{0,\alpha\beta}(\overline{\Omega})$. $DT(\nabla u_0)$ is uniformly strictly 2-elliptic according to the lower bound of condition (4). In addition $f \in C^{0,\alpha}(\overline{\Omega})$. Therefore according to [46] Thm 8.34, problem (4.5.2) admits a unique solution $w_0 \in C^{1,\tilde\beta}(\overline{\Omega})$ with $\tilde\beta = \min(\alpha, \alpha\beta) > 0$. According to the weak maximum principle, [46] Cor 8.2, it follows that $v_0 - w_0 = 0$. Thus v_0 in $C^{1,\tilde\beta}(\overline{\Omega})$. As Ω is bounded and as by definition $v_0 \in \mathcal{H}$, it follows $v_0 \in L^\infty(\Omega)$, $\nabla v_0 \in L^\infty(\Omega)$ and $v_0 \in \mathcal{V}$. □

Even when $\mathcal{J}_\varepsilon$ is not the compliance, we shall make the assumptions that $v_0 \in L^\infty(\Omega)$ and that $\nabla v_0 \in L^\infty(\Omega)$. As by definition $v_0 \in \mathcal{H}$, it follows that $v_0 \in \mathcal{V}$.

4.5.2 Step 1: variation $v_\varepsilon - v_0$

Let $\tilde v_\varepsilon = v_\varepsilon - v_0$. After (4.5.1), for all $\eta \in \mathcal{H}$, it holds:

$$\begin{aligned} 0 &= \int_\Omega \left[\gamma_\varepsilon DT(\nabla u_0)\nabla v_\varepsilon - \gamma_0 DT(\nabla u_0)\nabla v_0\right].\nabla\eta \\ &= \int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde v_\varepsilon.\nabla\eta + (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} DT(\nabla u_0)\nabla v_0.\nabla\eta. \end{aligned} \tag{4.5.3}$$

Hence

$$\int_{\Omega} \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde{v}_\varepsilon.\nabla\eta = -(\gamma_1-\gamma_0)\int_{\omega_\varepsilon} DT(\nabla u_0)\nabla v_0.\nabla\eta, \quad \forall\ \eta \in \mathcal{H}. \tag{4.5.4}$$

4.5.3 Step 2: approximation of variation $v_\varepsilon - v_0$

After condition (4), matrix $DT(U_0)$ is definite positive. Applying the Lax-Milgram theorem, we approximate the variation $\tilde{v}_\varepsilon$ by the unique function $k_\varepsilon \in \mathcal{H}$ such that

$$\int_{\Omega} \gamma_\varepsilon DT(U_0)\nabla k_\varepsilon.\nabla\eta = -(\gamma_1-\gamma_0)\int_{\omega_\varepsilon} DT(U_0)V_0.\nabla\eta, \quad \forall\eta \in \mathcal{H}. \tag{4.5.5}$$

4.5.4 Step 3: change of scale

For the purpose of defining the variation of the adjoint state at scale 1 in $\mathbb{R}^N$, we look for a function K which may approximate the map $y \in \Omega/\varepsilon \mapsto \varepsilon^{-1}k_\varepsilon(\varepsilon y)$. The weighted quotient Hilbert space $\mathcal{H}(\mathbb{R}^N)$ was defined in section 3.3 on page 43. An equivalence holds between the norm and the semi-norm in $\mathcal{H}(\mathbb{R}^n)$ as well as a coercivity property, as stated in Corollary 3.6.1 on page 50.Therefore, applying the Lax-Milgram theorem in $\mathcal{H}(\mathbb{R}^N)$, one obtains

Lemma 4.5.2. *There exists a unique function $K \in \mathcal{H}(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} \gamma DT(U_0)\nabla K.\nabla\eta = -(\gamma_1-\gamma_0)\int_{\omega} DT(U_0)V_0.\nabla\eta, \quad \forall\ \eta \in \mathcal{H}(\mathbb{R}^N). \tag{4.5.6}$$

4.5.5 Step 4: asymptotic behavior of variations of the adjoint state

For all $\varepsilon > 0$ small enough, let

$$K_\varepsilon : x \in \Omega \mapsto K_\varepsilon(x) := \varepsilon\hat{K}(\varepsilon^{-1}x) \tag{4.5.7}$$

where $\hat{K} \in \mathcal{H}^w(\mathbb{R}^N)$ denotes a given element of the class $K \in \mathcal{H}(\mathbb{R}^N)$.

Making the change of scale backward , as

$$\inf_{x\in\Omega} w_2\left(\frac{x}{\varepsilon}\right) > 0,$$

it follows from $\hat{K} \in \mathcal{H}^w(\mathbb{R}^N)$ that $K_\varepsilon \in H^1(\Omega)$.

Lemma 4.5.3. *It holds*

$$\begin{aligned} \|\nabla\tilde{v}_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N), && (4.5.8)\\ \|\nabla k_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N), && (4.5.9)\\ \|\nabla K_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N). && (4.5.10) \end{aligned}$$

The proof is available in subsection 4.7.3 on page 91.

Proposition 4.5.4. *There exists an element $\tilde{K}$ of the class $K \in \mathcal{H}(\mathbb{R}^N)$ such that*

$$\tilde{K}(y) = O\left(|y|^{1-N}\right) \quad as \quad |y| \to +\infty, \tag{4.5.11}$$

and

$$\nabla K(y) = O\left(|y|^{-N}\right) \quad as \quad |y| \to +\infty. \tag{4.5.12}$$

Moreover

$$K \in \mathcal{V}(\mathbb{R}^N). \tag{4.5.13}$$

The proof is available in subsection 4.7.3 on page 92.

From now on, function K_ε is defined choosing $\hat{K} = \tilde{K}$ in (4.5.7), i.e.

$$K_\varepsilon(x) := \varepsilon \tilde{K}(\varepsilon^{-1}x), \quad \forall x \in \Omega. \tag{4.5.14}$$

Lemma 4.5.5. *It holds*

$$\|\nabla k_\varepsilon - \nabla K_\varepsilon\|^2_{L^2(\Omega)} = o(\varepsilon^N), \tag{4.5.15}$$

and

$$\forall \alpha > 0, \forall r \in (0,1), \quad \int_{\Omega \setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon|^2 = o(\varepsilon^N). \tag{4.5.16}$$

The proof is available in subsection 4.7.3 on page 94.

Lemma 4.5.6. *It holds*

$$\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|^2_{L^2(\Omega)} = o(\varepsilon^N). \tag{4.5.17}$$

The proof is available in subsection 4.7.3 on page 96. Estimates stated in Lemmas 4.5.5 and 4.5.6 will be applied to obtain the topological asymptotic expansion in subsection 4.7.4.

4.6 Topological asymptotic expansion

For simplicity we denote

$$j(\varepsilon) := \mathcal{J}_\varepsilon(u_\varepsilon), \quad \forall \varepsilon \geq 0 \text{ small enough.} \tag{4.6.1}$$

Expansion (4.3.1) reads

$$j(\varepsilon) - j(0) = \langle G, \tilde{u}_\varepsilon \rangle + \delta_2\, \varepsilon^N + R(\varepsilon).$$

– In the first case (4.3.2), after estimate (4.4.9), it holds

$$R(\varepsilon) = o\left(\|u_\varepsilon - u_0\|^2_{\mathcal{H}}\right) = o\left(\varepsilon^N\right).$$

– In the second case (4.3.3), after estimate (4.4.31), it holds

$$R(\varepsilon) = O\left(\int_{\Omega\setminus B(0,\tilde{\alpha}\varepsilon^{\tilde{r}})} |\nabla \tilde{u}_\varepsilon|^p + |\nabla \tilde{u}_\varepsilon|^2\right) = o\left(\varepsilon^N\right).$$

Therefore

$$j(\varepsilon) - j(0) = \langle G, \tilde{u}_\varepsilon\rangle + \delta_2\,\varepsilon^N + o\left(\varepsilon^N\right).$$

Then plugging test function $\eta = \tilde{u}_\varepsilon \in \mathcal{V} \subset \mathcal{H}$ in variational form (4.5.1), one obtains

$$\begin{aligned} j(\varepsilon) - j(0) &= -\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde{u}_\varepsilon.\nabla v_\varepsilon + \delta_2\,\varepsilon^N + o(\varepsilon^N) \\ &= -\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde{u}_\varepsilon.\nabla v_0 - \int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde{u}_\varepsilon.\nabla\tilde{v}_\varepsilon + \delta_2\,\varepsilon^N + o(\varepsilon^N). \end{aligned} \tag{4.6.2}$$

One can plug $\eta = \tilde{u}_\varepsilon \in \mathcal{V} \subset \mathcal{H}$ in variational form (4.5.4) and obtains

$$\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde{v}_\varepsilon.\nabla\tilde{u}_\varepsilon + (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} DT(\nabla u_0)\nabla v_0.\nabla\tilde{u}_\varepsilon = 0. \tag{4.6.3}$$

After assumption made that $v_0 \in \mathcal{V}$ (see Lemma 4.5.1), one can plug $\eta = v_0$ in variational form (4.4.3) and obtains

$$\int_\Omega \gamma_\varepsilon \left[T(\nabla u_0 + \nabla\tilde{u}_\varepsilon) - T(\nabla u_0)\right].\nabla v_0 + (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(\nabla u_0).\nabla v_0 = 0. \tag{4.6.4}$$

Summing equalities (4.6.2), (4.6.3) and (4.6.4), it follows

$$j(\varepsilon) - j(0) = j_1(\varepsilon) + j_2(\varepsilon) + \delta_2\,\varepsilon^N + o(\varepsilon^N) \tag{4.6.5}$$

with

$$j_1(\varepsilon) := (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(\nabla u_0).\nabla v_\varepsilon \tag{4.6.6}$$

and

$$\begin{aligned} j_2(\varepsilon) := &\int_\Omega \gamma_\varepsilon \left[T(\nabla\tilde{u}_\varepsilon + \nabla u_0) - T(\nabla u_0) - DT(\nabla u_0)(\nabla\tilde{u}_\varepsilon)\right].\nabla v_0 \\ &+ (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} \left[DT(\nabla u_0)\nabla v_0.\nabla\tilde{u}_\varepsilon - T(\nabla u_0).\nabla\tilde{v}_\varepsilon\right] \\ = &\int_\Omega \gamma_\varepsilon S_{\nabla u_0}(\nabla\tilde{u}_\varepsilon).\nabla v_0 + (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} \left[DT(\nabla u_0)\nabla v_0.\nabla\tilde{u}_\varepsilon - T(\nabla u_0).\nabla\tilde{v}_\varepsilon\right]. \end{aligned} \tag{4.6.7}$$

As mentioned in section 4.1, the map S_φ accounts for the nonlinearity of gradient field T, that is to the non-quadratic behavior of W at point φ. In particular, if $p = 2$ then $S_\varphi = 0,\ \forall\varphi \in \mathbb{R}^N$.

We shall see hereinafter that $j_2(\varepsilon)$ accounts for the contribution of the nonlinear behavior of T to $j(\varepsilon)$, while $j_1(\varepsilon)$ provides the variation of $j(\varepsilon)$ caused by the affine component of T.

4.6.1 Expansion of linear term $j_1(\varepsilon)$

Following approximation steps 2 and 3, we define

$$\tilde{j}_1(\varepsilon) \; := \; (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} T(U_0).(V_0 + \nabla k_\varepsilon), \tag{4.6.8}$$

and

$$\begin{aligned} J_1 \; &:= \; (\gamma_1 - \gamma_0) \int_{\omega} T(U_0).(V_0 + \nabla K) \\ &= \; (\gamma_1 - \gamma_0) \; T(U_0). \left[|\omega| \, V_0 + \int_{\partial\omega} K n_{out} \right], \end{aligned} \tag{4.6.9}$$

the last equality after Green's formula, where n_{out} denotes the unit outward normal to $\partial\omega$.

Regarding the calculation of the latter integral, it follows from the linearity of equation (4.5.6) that the mapping

$$V_0 \mapsto (\gamma_1 - \gamma_0) \left[|\omega| \, V_0 + \int_{\partial\omega} K n_{out} \right]$$

is linear $\mathbb{R}^N \to \mathbb{R}^N$. It only depends on the set ω, on the definite positive matrix $DT(U_0)$ and on the ratio γ_1/γ_0. Hence there exists a polarization tensor

$$\mathcal{P} = \mathcal{P}(\omega, DT(U_0), \gamma_1/\gamma_0),$$

such that

$$(\gamma_1 - \gamma_0) \left[|\omega| \, V_0 + \int_{\partial\omega} K n_{out} \right] \; = \; \mathcal{P} V_0 \tag{4.6.10}$$

(see e.g. [74, 34, 5, 12]). Eventually it follows

$$J_1 = T(U_0).\,(\mathcal{P} V_0) = T(U_0)^T \mathcal{P} V_0. \tag{4.6.11}$$

Lemma 4.6.1. *For all $\varepsilon \geq 0$ small enough it holds*

$$\tilde{j}_1(\varepsilon) - \varepsilon^N J_1 = o(\varepsilon^N). \tag{4.6.12}$$

The proof is available in subsection 4.7.4 on page 98.

Lemma 4.6.2. *For all $\varepsilon \geq 0$ small enough it holds*

$$j_1(\varepsilon) - \tilde{j}_1(\varepsilon) = o(\varepsilon^N). \tag{4.6.13}$$

The proof is available in subsection 4.7.4 on page 98.

Summing estimates (4.6.12) and (4.6.13) it follows that

Proposition 4.6.3.

$$j_1(\varepsilon) = \varepsilon^N T(U_0)^T \mathcal{P} V_0 + \; o\left(\varepsilon^N\right). \tag{4.6.14}$$

4.6.2 Expansion of nonlinear term $j_2(\varepsilon)$

The term $j_2(\varepsilon)$ can be approximated according to approximation steps 2 and 3 as follows. Since $\nabla h_\varepsilon \in L^p(\Omega)$ and $\nabla k_\varepsilon \in L^2(\Omega)$, after growth condition (4.1.2) one can define

$$\tilde{j}_2(\varepsilon) := \int_\Omega \gamma_\varepsilon S_{U_0}(\nabla h_\varepsilon).V_0 + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} \left[DT(U_0)V_0.\nabla h_\varepsilon - T(U_0).\nabla k_\varepsilon\right]. \quad (4.6.15)$$

Similarly since $\nabla H \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and $\nabla K \in L^2(\mathbb{R}^N)$, after growth condition (4.1.2) one can define

$$J_2 := \int_{\mathbb{R}^N} \gamma S_{U_0}(\nabla H).V_0 + (\gamma_1 - \gamma_0) \int_\omega \left[DT(U_0)V_0.\nabla H - T(U_0).\nabla K\right]. \quad (4.6.16)$$

In addition after Propositions 4.4.11 and 4.5.4, it holds $H \in \mathcal{H}(\mathbb{R}^N)$ and $K \in \mathcal{V}(\mathbb{R}^N)$. Plugging function test K into variational form (4.4.5) defining H and plugging function test H into variational form(4.5.6) defining K, one obtains

$$J_2 = \int_{\mathbb{R}^N} \gamma S_{U_0}(\nabla H).(V_0 + \nabla K). \quad (4.6.17)$$

Lemma 4.6.4. *For $\varepsilon \geq 0$ small enough it holds*

$$\tilde{j}_2(\varepsilon) - \varepsilon^N J_2 = o(\varepsilon^N). \quad (4.6.18)$$

The proof is available in subsection 4.7.4 on page 99.

Lemma 4.6.5. *It holds*

$$\int_\Omega |\nabla v_0 - V_0| \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) = o(\varepsilon^N), \quad (4.6.19)$$

$$\forall\, p \in (3, \infty), \int_\Omega |\nabla v_0 - V_0|\, |\nabla h_\varepsilon|^{p-1} = o(\varepsilon^N). \quad (4.6.20)$$

The proof is available in subsection 4.7.4 on page 100.

Lemma 4.6.6. *For $\varepsilon \geq 0$ small enough it holds*

$$j_2(\varepsilon) - \tilde{j}_2(\varepsilon) = o(\varepsilon^N). \quad (4.6.21)$$

The proof is available in subsection 4.7.4 on page 101.

Eventually summing estimates (4.6.18) and (4.6.21) yields

Proposition 4.6.7.

$$j_2(\varepsilon) = \varepsilon^N \left(\int_{\mathbb{R}^N} \gamma S_{U_0}(\nabla H).\,(V_0 + \nabla K)\right) + o\left(\varepsilon^N\right). \quad (4.6.22)$$

4.6.3 Topological asymptotic expansion

Lastly according to (4.6.5), summing the estimate of $j_1(\varepsilon)$ given by (4.6.14) and the estimate of $j_2(\varepsilon)$ given by (4.6.22), one obtains the topological asymptotic expansion claimed in Theorem 4.3.1 on page 56.

4.7 Proofs

4.7.1 Proofs about potential W_a

Proof of Proposition 4.1.2

Let $\varphi, \psi \in \mathbb{R}^N$. It is easy to check the following algebraic identity by expanding both sides

$$\begin{aligned} R(\varphi,\psi) &:= \left[(a^2+|\varphi+\psi|^2)^{\frac{p-2}{2}}(\varphi+\psi)-(a^2+|\varphi|^2)^{\frac{p-2}{2}}\varphi\right].\psi \\ &= \frac{(a^2+|\varphi+\psi|^2)^{\frac{p-2}{2}}-(a^2+|\varphi|^2)^{\frac{p-2}{2}}}{2}\left(|\varphi+\psi|^2-|\varphi|^2\right) \\ &\qquad + \frac{(a^2+|\varphi+\psi|^2)^{\frac{p-2}{2}}+(a^2+|\varphi|^2)^{\frac{p-2}{2}}}{2}\,|\psi|^2. \end{aligned}$$

By monotonicity of the function $x \in \mathbb{R}_+ \mapsto (a^2+x)^{\frac{p-2}{2}}$, the first term on the right-hand side is always non negative. Hence

$$R(\varphi,\psi) \geq \frac{(a^2+|\varphi+\psi|^2)^{\frac{p-2}{2}}+(a^2+|\varphi|^2)^{\frac{p-2}{2}}}{2}\,|\psi|^2. \tag{4.7.1}$$

It follows immediately from (4.7.1) that

$$R(\varphi,\psi) \geq c_2\,|\psi|^2,$$

where $c_2 := a^{p-2} > 0$ does not depend on φ, ψ.

Let us now prove the p-coercivity. As the case $\psi = 0$ is trivial, we assume $\psi \neq 0$ and we decompose φ as follows

$$\varphi = \xi + s\psi, \text{ with } \xi \in \mathbb{R}^N, s \in \mathbb{R} \text{ and } \xi.\psi = 0.$$

Let $b \in \mathbb{R}_+$ such that $b^2 = a^2 + |\xi|^2$. The Pythagorean theorem yields

$$R(\varphi,\psi) = \left[\left(b^2+(s+1)^2|\psi|^2\right)^{\frac{p-2}{2}}(s+1)-\left(b^2+s^2|\psi|^2\right)^{\frac{p-2}{2}}s\right]|\psi|^2.$$

Let

$$d := \frac{b}{|\psi|} \quad \text{and} \quad Q(\varphi,\psi) := \frac{R(\varphi,\psi)}{|\psi|^p}.$$

Thus

$$Q(\varphi,\psi) = \left(d^2+(s+1)^2\right)^{\frac{p-2}{2}}(s+1)-\left(d^2+s^2\right)^{\frac{p-2}{2}}s. \tag{4.7.2}$$

We distinguish between several cases.

1. If $s \geq 0$. Due to the monotonicity of the map $x \in \mathbb{R}_+ \mapsto (d^2+x^2)^{\frac{p-2}{2}}$, it holds

$$\left[\left(d^2+(s+1)^2\right)^{\frac{p-2}{2}}-\left(d^2+s^2\right)^{\frac{p-2}{2}}\right]s \geq 0$$

and

$$\left(d^2+(s+1)^2\right)^{\frac{p-2}{2}} \geq \left(d^2+1\right)^{\frac{p-2}{2}} \geq 1.$$

It thus follows from formula (4.7.2) that $Q(\varphi,\psi) \geq 1$.

2. If $s < 0$. Let $t := |s| = -s$. Rewrite (4.7.2) as follows

$$Q(\varphi,\psi) = \left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} (1-t) + \left(d^2 + t^2\right)^{\frac{p-2}{2}} t. \tag{4.7.3}$$

We distinguish again two cases.

(a) If $t \le 1$. Due to the monotonicity of the maps $x \in \mathbb{R}_+ \mapsto x^{\frac{p-2}{2}}$ and $x \in \mathbb{R}_+ \mapsto x^{p-1}$ and since

$$\max(1-x, x) \ge \frac{1}{2}, \quad \forall x \in (0,1],$$

one obtains from (4.7.3) that

$$Q(\varphi,\psi) \ge (1-t)^{p-2}(1-t) + t^{p-2}t = (1-t)^{p-1} + t^{p-1} \ge 2^{1-p}.$$

(b) If $t > 1$. As $0 < t-1 < t$, by monotonicity, it holds

$$-\left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} + \left(b^2 + t^2\right)^{\frac{p-2}{2}} \ge 0.$$

From equation (4.7.3), one thus obtains

$$\begin{aligned} Q(\varphi,\psi) &= \left[-\left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} + \left(d^2 + t^2\right)^{\frac{p-2}{2}}\right] t + \left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} \\ &\ge \left[-\left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} + \left(d^2 + t^2\right)^{\frac{p-2}{2}}\right] + \left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} \\ &= \left(d^2 + t^2\right)^{\frac{p-2}{2}} \ge \left(d^2 + 1\right)^{\frac{p-2}{2}} \ge 1. \end{aligned}$$

Let $c_p = \min(1, 2^{1-p}) = 2^{1-p}$. We have thus proved that

$$R(\varphi,\psi) \ge c_p \, |\psi|^p, \quad \forall\, \varphi,\psi \in \mathbb{R}^N.$$

Lastly choosing $c = \frac{1}{2}\min(c_2, c_p)$ completes the proof of Proposition 4.1.2.

Proof of Proposition 4.1.3

Let us prove that for all $a > 0$, the potential W_a satisfies Assumption 4.1.1.

1. The two maps $\varphi \in \mathbb{R}^N \mapsto |\varphi|^2$ and $t \in \mathbb{R}_+ \mapsto \frac{1}{p}(a^2 + t)^{\frac{p}{2}}$ are C^∞. It follows that the composite function $W_a \in C^\infty(\mathbb{R}^N, \mathbb{R})$.
2. Regarding condition (2), it is obvious that the lower bound holds with $a_0 := \frac{1}{p}$. Then since $p/2 \ge 1$, the function $\lambda \in \mathbb{R}_+ \mapsto \lambda^{\frac{p}{2}}$ is convex. Thus for all $\varphi \in \mathbb{R}^N$ it follows by convexity that

$$W_a(\varphi) = \frac{1}{p}\left(a^2 + |\varphi|^2\right)^{\frac{p}{2}} \le \frac{1}{p} 2^{\frac{p-2}{2}} \left(a^p + |\varphi|^p\right).$$

Hence the upper bound of condition (2) holds choosing $b_0 := \frac{1}{p} 2^{\frac{p-2}{2}} \max(a^p, 1)$.

3. About condition (3), it first holds
$$T_a(\varphi) = \left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}} \varphi \quad \forall \varphi \in \mathbb{R}^N.$$
Thus
$$\begin{cases} |T_a(\varphi)| \leq 2^{\frac{p-2}{2}} a^{p-2} |\varphi| & \text{if } |\varphi| \leq a, \\ |T_a(\varphi)| \leq 2^{\frac{p-2}{2}} |\varphi|^{p-1} & \text{if } |\varphi| > a. \end{cases}$$
Hence inequality in condition (3) holds choosing $a_1 := 2^{\frac{p-2}{2}} \max(a^{p-2}, 1)$.
4. For all $\varphi, \psi \in \mathbb{R}^N$ it holds
$$DT_a(\varphi)\psi = (p-2)\left(a^2 + |\varphi|^2\right)^{\frac{p-4}{2}} (\varphi.\psi)\varphi + \left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}} \psi.$$
Thus
$$\begin{aligned} DT_a(\varphi)\psi.\psi &= (p-2)\left(a^2 + |\varphi|^2\right)^{\frac{p-4}{2}} (\varphi.\psi)^2 + \left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}} |\psi|^2 \\ &\geq \left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}} |\psi|^2. \end{aligned}$$
Hence the lower bound in condition (4) holds choosing $c := \min(1, a^{p-2}) > 0$. Moreover after Cauchy-Schwarz's inequality it holds
$$\begin{aligned} DT_a(\varphi)\psi.\psi &= (p-2)\left(a^2 + |\varphi|^2\right)^{\frac{p-4}{2}} (\varphi.\psi)^2 + \left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}} |\psi|^2 \\ &\leq \left[(p-2)\left(a^2 + |\varphi|^2\right)^{\frac{p-4}{2}} |\varphi|^2 + \left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}}\right] |\psi|^2 \\ &\leq (p-1)\left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}} |\psi|^2. \end{aligned}$$
Hence the upper bound in condition (4) holds choosing $C := (p-1)\max(1, a^{p-2})$.
5. Condition (5) follows immediately from Proposition 4.1.2.
6. Regarding condition (6), let $\varphi, \psi \in \mathbb{R}^N$. Let $g : t \in (0,1) \mapsto T_a(\varphi + t\psi)$. The equality
$$g(1) - g(0) = \int_0^1 g'(t) \mathrm{d}t$$
may be expanded into
$$\begin{aligned} T_a(\varphi + \psi) - T_a(\varphi) &= (p-2)\int_0^1 \left(a^2 + |\varphi + t\psi|^2\right)^{\frac{p-4}{2}} ((\varphi + t\psi).\psi)(\varphi + t\psi)\ \mathrm{d}t \\ &\quad + \int_0^1 \left(a^2 + |\varphi + t\psi|^2\right)^{\frac{p-2}{2}} \psi\ \mathrm{d}t. \end{aligned}$$
Thus
$$\begin{aligned} |T_a(\varphi + \psi) - T_a(\varphi)| &\leq (p-2)\int_0^1 \left(a^2 + |\varphi + t\psi|^2\right)^{\frac{p-4}{2}} |\varphi + t\psi|^2 |\psi|\ \mathrm{d}t \\ &\quad + \int_0^1 \left(a^2 + |\varphi + t\psi|^2\right)^{\frac{p-2}{2}} |\psi|\ \mathrm{d}t \\ &\leq (p-1)|\psi| \int_0^1 \left(a^2 + |\varphi + t\psi|^2\right)^{\frac{p-2}{2}}\ \mathrm{d}t \\ &\leq (p-1)|\psi| \left(a^2 + 2|\varphi|^2 + 2|\psi|^2\right)^{\frac{p-2}{2}}. \end{aligned} \tag{4.7.4}$$

Moreover

$$\begin{cases} \left(a^2 + 2\left|\varphi\right|^2 + 2\left|\psi\right|^2\right)^{\frac{p-2}{2}} \leq 2^{\frac{p-2}{2}} \left(a^2 + 2\left|\varphi\right|^2\right)^{\frac{p-2}{2}} & \text{if } 2\left|\psi\right|^2 \leq a^2 + 2\left|\varphi\right|^2, \\ \left(a^2 + 2\left|\varphi\right|^2 + 2\left|\psi\right|^2\right)^{\frac{p-2}{2}} \leq 2^{p-2}\left|\psi\right|^{p-2} & \text{if } 2\left|\psi\right|^2 > a^2 + 2\left|\varphi\right|^2. \end{cases}$$

Hence inequality (4.7.4) entails

$$\begin{aligned} |T_a(\varphi+\psi) - T_a(\varphi)| &\leq (p-1)\left|\psi\right| \left[2^{\frac{p-2}{2}} \left(a^2 + 2\left|\varphi\right|^2\right)^{\frac{p-2}{2}} + 2^{p-2}\left|\psi\right|^{p-2}\right] \\ &\leq (p-1)\left|\psi\right| \left[2^{p-2}a^{p-2} + 2^{\frac{3(p-2)}{2}}\left|\varphi\right|^{p-2} + 2^{p-2}\left|\psi\right|^{p-2}\right]. \end{aligned}$$

Let $C := (p-1)2^{p-2}\max(a^{p-2}, 2^{\frac{p-2}{2}})$. We thus have

$$|T_a(\varphi+\psi) - T_a(\varphi)| \leq C\left|\psi\right| \left[1 + \left|\varphi\right|^{p-2} + \left|\psi\right|^{p-2}\right] \quad \forall \varphi, \psi \in \mathbb{R}^N$$

which completes the proof of condition (6).

7. Regarding condition (7), let $M > 0$ and let $\varphi \in B(0, M)$. For all $\psi_1, \psi_2 \in \mathbb{R}^N$, the Taylor formula reads

$$\begin{aligned} &S_\varphi(\psi_2) - S_\varphi(\psi_1) = T_a(\varphi+\psi_2) - T_a(\varphi+\psi_1) - DT_a(\varphi)(\psi_2 - \psi_1) \\ &= \int_0^1 \left[DT_a(\varphi + \psi_1 + t(\psi_2 - \psi_1)) - DT_a(\varphi)\right](\psi_2 - \psi_1))\, dt \\ &= \int_0^1\int_0^1 D^2T_a\left(\varphi + s\left[(1-t)\psi_1 + t\psi_2\right]\right)\left((1-t)\psi_1 + t\psi_2\right)(\psi_2 - \psi_1)\, ds\, dt. \end{aligned} \tag{4.7.5}$$

Recall

$$T_a(\xi_1) = \left(a^2 + \left|\xi_1\right|^2\right)^{\frac{p-2}{2}} \xi_1, \quad \forall \xi_1 \in \mathbb{R}^N.$$

For all $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^N$ one obtains by differentiation

$$DT_a(\xi_1)(\xi_2) = (p-2)\left(a^2 + \left|\xi_1\right|^2\right)^{\frac{p-4}{2}} (\xi_1.\xi_2)\xi_1 + \left(a^2 + \left|\xi_1\right|^2\right)^{\frac{p-2}{2}} \xi_2.$$

Then

$$\begin{aligned} D^2T_a(\xi_1)(\xi_2, \xi_3) &= (p-2)(p-4)\left(a^2 + \left|\xi_1\right|^2\right)^{\frac{p-6}{2}} (\xi_1.\xi_2)(\xi_1.\xi_3)\xi_1 \\ &+ (p-2)\left(a^2 + \left|\xi_1\right|^2\right)^{\frac{p-4}{2}} \left[(\xi_2.\xi_3)\xi_1 + (\xi_1.\xi_2)\xi_3 + (\xi_1.\xi_3)\xi_2\right]. \end{aligned} \tag{4.7.6}$$

After Cauchy-Schwarz's inequality it follows

$$\left|D^2T_a(\xi_1)(\xi_2, \xi_3)\right| \leq C(p)\left(a^2 + \left|\xi_1\right|^2\right)^{\frac{p-3}{2}} \left|\xi_2\right|\left|\xi_3\right|.$$

where $C(p) = (p-2)(|p-4|+3)$. In particular the case $p = 2$ implies $C(p) = 0$.

– If $p \in [2, 3]$ then

$$\left|D^2T_a(\xi_1)(\xi_2, \xi_3)\right| \leq C(p)a^{p-3}\left|\xi_2\right|\left|\xi_3\right|.$$

Hence Taylor formula (4.7.5) entails

$$|S_\varphi(\psi_2) - S_\varphi(\psi_1)| \leq C(p)a^{p-3}\left|\psi_2 - \psi_1\right|(\left|\psi_1\right| + \left|\psi_2\right|).$$

Therefore condition (7) holds with $c_0 = C(p)a^{p-3}$ and $c_{p-3} = 0$. We check that $c_0 = 0$ if $p = 2$.

– If $p \in (3,\infty)$, for all $s,t \in (0,1)$ it holds

$$\left(a^2 + |\varphi + s\,[(1-t)\psi_1 + t\psi_2]|^2\right)^{\frac{p-3}{2}} \le \left(a^2 + 2\,|\varphi|^2 + 2\,(|\psi_1| + |\psi_2|)^2\right)^{\frac{p-3}{2}}$$
$$\le 2^{\frac{p-3}{2}}\left(a^2 + 2M^2\right)^{\frac{p-3}{2}} + 2^{p-3}\,(|\psi_1| + |\psi_2|)^{p-3}.$$

Therefore the Taylor formula (4.7.5) yields condition (7)

$$|S_\varphi(\psi_2) - S_\varphi(\psi_1)| \le |\psi_2 - \psi_1|\,(|\psi_1| + |\psi_2|)\left[c_0 + c_{p-3}\,(|\psi_1| + |\psi_2|)^{p-3}\right]$$

with $c_0 = 2^{\frac{p-3}{2}} C(p)\,(a^2 + 2M^2)^{\frac{p-3}{2}}$ and $c_{p-3} = 2^{p-3}C(p)$.

8. Regarding condition (8), we introduce for the sake of clarity operator Z defined by

$$Z_\psi(\varphi) := S_\varphi(\psi) = T_a(\varphi + \psi) - T_a(\varphi) - DT_a(\varphi)(\psi), \quad \forall \varphi, \psi \in \mathbb{R}^N.$$

For a given $\psi \in \mathbb{R}^N$, the map $\varphi \mapsto Z_\psi(\varphi)$ is C^∞. According to the Taylor formula with integral remainder, for all $\varphi, \psi, \xi \in \mathbb{R}^N$ it holds

$$\begin{aligned} DZ_\psi(\varphi)(\xi) &= DT_a(\varphi+\psi)(\xi) - DT_a(\varphi)(\xi) - D^2T_a(\varphi)(\psi,\xi) \\ &= \int_0^1 (1-s)D^3T_a(\varphi + s\psi)(\psi,\xi,\psi)\,ds. \end{aligned}$$

Let $M > 0$ and let $\varphi_1, \varphi_2 \in B(0,M)$. For all $\psi \in \mathbb{R}^N$ it thus holds

$$S_{\varphi_2}(\psi) - S_{\varphi_1}(\psi) = Z_\psi(\varphi_2) - Z_\psi(\varphi_1) = \int_0^1 DZ_\psi(\varphi_1 + t(\varphi_2 - \varphi_1))(\varphi_2 - \varphi_1)\,dt$$
$$= \int_0^1\int_0^1 (1-s)D^3T_a(\varphi_1 + t(\varphi_2-\varphi_1) + s\psi)(\psi, \varphi_2 - \varphi_1, \psi)\,ds\,dt. \quad (4.7.7)$$

Differentiating (4.7.6), for all $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^N$ one obtains

$$\begin{aligned} &D^3T_a(\xi_1)(\xi_2,\xi_3,\xi_2) = \\ &(p-2)(p-4)(p-6)\left(a^2 + |\xi_1|^2\right)^{\frac{p-8}{2}}(\xi_1.\xi_2)^2(\xi_1.\xi_3)\xi_1 \\ &+ (p-2)(p-4)\left(a^2 + |\xi_1|^2\right)^{\frac{p-6}{2}}|\xi_2|^2\,(\xi_1.\xi_3)\xi_1 \\ &+ (p-2)(p-4)\left(a^2 + |\xi_1|^2\right)^{\frac{p-6}{2}}(\xi_1.\xi_2)(\xi_2.\xi_3)\xi_1 \\ &+ (p-2)(p-4)\left(a^2 + |\xi_1|^2\right)^{\frac{p-6}{2}}(\xi_1.\xi_2)(\xi_1.\xi_3)\xi_2 \\ &+ (p-2)(p-4)\left(a^2 + |\xi_1|^2\right)^{\frac{p-6}{2}}(\xi_1.\xi_2)\left[(\xi_2.\xi_3)\xi_1 + (\xi_1.\xi_2)\xi_3 + (\xi_1.\xi_3)\xi_2\right] \\ &\qquad + (p-2)\left(a^2 + |\xi_1|^2\right)^{\frac{p-4}{2}}\left[|\xi_2|^2\,\xi_3 + 2(\xi_2.\xi_3)\xi_2\right]. \end{aligned}$$

Thus

$$\left|D^3T_a(\xi_1)(\xi_2,\xi_3,\xi_2)\right| \le C(p)\left(a^2 + |\xi_1|^2\right)^{\frac{p-4}{2}}|\xi_2|^2\,|\xi_3|.$$

where $C(p) = (p-2)[3 + |p-4|\,(6 + |p-6|)]$. In particular the case $p = 2$ implies $C(p) = 0$.

– If $p \in [2,4]$, it holds

$$\left| D^3 T_a(\xi_1)(\xi_2,\xi_2,\xi_3) \right| \leq C(p) a^{p-4} \left|\xi_2\right|^2 \left|\xi_3\right|.$$

Hence it follows from (4.7.7)

$$\left|S_{\varphi_2}(\psi) - S_{\varphi_1}(\psi)\right| \leq \frac{1}{2} C(p) a^{p-4} \left|\varphi_2 - \varphi_1\right| \left|\psi\right|^2.$$

Therefore condition (8) holds with $d_0 = C(p)a^{p-4}/2$ and $d_{p-4} = 0$. We check that $d_0 = 0$ if $p = 2$.

– If $p > 4$, for all $t, s \in (0,1)$ it holds

$$\begin{aligned}\left(a^2 + \left|\varphi_1 + t(\varphi_2 - \varphi_1) + s\psi\right|^2\right)^{\frac{p-4}{2}} &\leq \left(a^2 + 2M^2 + 2\left|\psi\right|^2\right)^{\frac{p-4}{2}} \\ &\leq 2^{\frac{p-4}{2}} \left(a^2 + 2M^2\right)^{\frac{p-4}{2}} + 2^{p-4} \left|\psi\right|^{p-4}.\end{aligned}$$

Hence the Taylor formula (4.7.7) yields

$$\left|S_{\varphi_2}(\psi) - S_{\varphi_2}(\psi)\right| \leq \left|\varphi_2 - \varphi_1\right| \left(d_0 \left|\psi\right|^2 + d_{p-4} \left|\psi\right|^{p-2}\right)$$

with $d_0(M,p) = 2^{\frac{p-6}{2}} C(p) \left(a^2 + 2M^2\right)^{\frac{p-4}{2}}$ and $d_{p-4}(p) = 2^{p-5} C(p)$.

This completes the proof of condition (8).

4.7.2 Proofs about the variation of the direct state

Proof of Lemma 4.2.1

It follows from the upper-bound of condition (2) and from $f \in C^{0,\alpha}(\overline{\Omega}) \subset L^q(\Omega)$ that the functional $\mathcal{W}_\varepsilon$ is well defined in $\mathcal{V}$.

1. Applying the Lebesgue dominated convergence theorem, it is standard to prove that functional $\mathcal{W}_\varepsilon$ is Fréchet differentiable in $\mathcal{V}$ and that

$$D\mathcal{W}_\varepsilon(u)(\eta) = \int_\Omega \left[\gamma_\varepsilon T(\nabla u).\nabla \eta - f\eta\right], \quad \forall u, \eta \in \mathcal{V}. \tag{4.7.8}$$

   See e.g. [17], proof of Thm. 6.6.1. Note that according to condition (3), $\nabla u \in L^p(\Omega)$ implies that $T(\nabla u) \in L^q(\Omega)$. Hence the integral in (4.7.8) is well defined.
2. The strict convexity of functional $\mathcal{W}_\varepsilon$ follows immediately from that stated for potential W.
3. After Poincaré inequality in $\mathcal{V}$ and after condition (2), it holds

$$\mathcal{W}_\varepsilon(u) \geq \underline{\gamma} a_0 |u|_{\mathcal{V}}^p - C \left\|f\right\|_{L^q(\Omega)} \left|u\right|_{\mathcal{V}}, \quad \forall u \in \mathcal{V},$$

   which entails the coercivity of $\mathcal{W}_\varepsilon$ in $\mathcal{V}$.

Therefore (see e.g. [17], Theorem 3.3.4.) the minimization problem of $\mathcal{W}_\varepsilon$ in $\mathcal{V}$ admits a unique solution

$$\{u_\varepsilon\} = \operatorname*{argmin}_{\eta \in \mathcal{V}} \mathcal{W}_\varepsilon(\eta).$$

This unique solution is equivalently defined by the first order condition $D\mathcal{W}_\varepsilon(u_\varepsilon) = 0$ which is the claimed Euler-Lagrange equation.

Proof of Lemma 4.4.5

1. Plugging $\eta = \tilde{u}_\varepsilon \in \mathcal{V}$ the variational form (4.4.3) yields
$$\int_\Omega \gamma_\varepsilon \left[T(\nabla\tilde{u}_\varepsilon + \nabla u_0) - T(\nabla u_0)\right].\nabla\tilde{u}_\varepsilon \;=\; -\,(\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(\nabla u_0).\nabla\tilde{u}_\varepsilon. \quad (4.7.9)$$
It follows from condition (5) that there exists $c > 0$ such that
$$\begin{aligned}&\underline{\gamma}\; c(\|\nabla\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega)})\\ &\qquad\le \int_\Omega \gamma_\varepsilon \left[T(\nabla\tilde{u}_\varepsilon + \nabla u_0) - T(\nabla u_0)\right].\nabla\tilde{u}_\varepsilon. \quad (4.7.10)\end{aligned}$$
In addition $\nabla u_0 \in L^\infty(\bar{\Omega})$ after Lemma 4.4.1 and T is continuous. Thus let
$$M := \sup\left\{|T(\psi)|\,;\,|\psi| \le \|\nabla u_0\|_{L^\infty(\Omega)}\right\} < \infty.$$
– According to Hölder's inequality it holds
$$\left|\int_{\omega_\varepsilon} T(\nabla u_0).\nabla\tilde{u}_\varepsilon\right| \le M\,|\omega|^{\frac{1}{q}}\,\varepsilon^{\frac{N}{q}}\,\|\nabla\tilde{u}_\varepsilon\|_{L^p(\Omega)}\,.$$
Therefore equations (4.7.9) and (4.7.10) imply
$$\underline{\gamma}\; c\,\|\nabla\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} \le |\gamma_1 - \gamma_0|\,M\,|\omega|^{\frac{1}{q}}\,\varepsilon^{\frac{N}{q}}\,\|\nabla\tilde{u}_\varepsilon\|_{L^p(\Omega)}\,.$$
Dividing both sides by $\|\nabla\tilde{u}_\varepsilon\|_{L^p(\Omega)}$ and powering the inequality to the power of q entails
$$\|\nabla\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} \;\le\; C\,\varepsilon^N = O(\varepsilon^N)$$
with $C := |\omega|\;\left[\;|\gamma_1 - \gamma_0|\;\;M\;/\;\underline{\gamma}c\,\right]^q$.
– Similarly applying Cauchy-Schwarz's inequality, it holds
$$\left|\int_{\omega_\varepsilon} T(\nabla u_0).\nabla\tilde{u}_\varepsilon\right| \le M\,|\omega|^{\frac{1}{2}}\,\varepsilon^{\frac{N}{2}}\,\|\nabla\tilde{u}_\varepsilon\|_{L^2(\Omega)}\,.$$
Hence one obtains from (4.7.9) and (4.7.10) that
$$\|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega)} = \;O(\varepsilon^N)$$
which completes the proof of the first claimed estimate (4.4.9).
2. The proof of estimate (4.4.10) is similar to the one of (4.4.9), starting from variational form (4.4.4).
3. Lastly, since $H \in \mathcal{V}(\mathbb{R}^N)$, by definition it holds $\nabla H \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Thus making a change of scale yields
$$\|\nabla H_\varepsilon\|^p_{L^p(\Omega)} = \int_\Omega |\nabla H_\varepsilon|^p = \varepsilon^N \;\int_{\Omega/\varepsilon} |\nabla H|^p \le \varepsilon^N\;\;\|\nabla H\|^p_{L^p(\Omega)} = O(\varepsilon^N).$$
Similarly one obtains
$$\|\nabla H_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N),$$
which completes the proof of estimate (4.4.11).

Remark 4.7.1. By convexity, it follows immediately from estimates (4.4.9), (4.4.10) and (4.4.11) that

$$\begin{aligned} \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N), && (4.7.11)\\ \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N). && (4.7.12) \end{aligned}$$

Moreover $\tilde{u}_\varepsilon, h_\varepsilon \in \mathcal{V} \subset \mathcal{H}$. Thus according to Poincaré inequalities in $\mathcal{V}$ and in $\mathcal{H}$, inequalities (4.4.9) and (4.4.10) imply

$$\begin{aligned} \|\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} + \|\tilde{u}_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N), && (4.7.13)\\ \|h_\varepsilon\|^p_{L^p(\Omega)} + \|h_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N). && (4.7.14) \end{aligned}$$

By convexity again it follows from (4.7.13) and (4.7.14) that

$$\|\tilde{u}_\varepsilon - h_\varepsilon\|^p_{L^p(\Omega)} + \|\tilde{u}_\varepsilon - h_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N). \tag{4.7.15}$$

Proof of Proposition 4.4.6

By assumption it holds $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ and $W = W_a$ for some $a > 0$. Moreover we assume that $p \in [2, \bar{p})$, where $\bar{p}$ is defined by equation (4.4.13) on page 62.

For all $\beta \in (N/2, N)$, let the function $P : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\begin{cases} P(x) := k(U_0.x)\,|x|^{-\beta}, & \text{if } |x| > 1,\\ P(x) := k(U_0.x), & \text{if } |x| \le 1, \end{cases} \tag{4.7.16}$$

where

$$k := \frac{\gamma_0 - \gamma_1}{\gamma_1 + \gamma_0(\beta - 1)}. \tag{4.7.17}$$

It is easy to check that $P \in \mathcal{V}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Recall we denote

$$\mathbb{R}^N_+ = \left\{ x \in \mathbb{R}^N;\ U_0.\ x \ge 0 \right\}.$$

Let us prove that there exists $\beta \in (N/2, N)$ such that P is a supersolution of operator Q in the half-space $\mathbb{R}^N_+$.

We shall need the following elementary inequalities: for all $\beta > N/2$, it holds

$$\begin{aligned} 1 + k(1-\beta) &> 0, && (4.7.18)\\ -2 + k(\beta - 2) &\le 0. && (4.7.19) \end{aligned}$$

According to the Green formula, one can split operator Q into the sum of three operators Q_{int}, Q_{trans} and Q_{ext} with supports respectively in $\bar{\omega}$, on $\partial\omega$ and in $\mathbb{R}^N \setminus \omega$ as follows:

$$\begin{aligned} \langle QP, \eta \rangle &:= \int_{\mathbb{R}^N} \gamma \left[T(U_0 + \nabla P) - T(U_0) \right].\nabla \eta + (\gamma_1 - \gamma_0) \int_\omega T(U_0).\nabla \eta,\\ &= \langle Q_{int} P, \eta \rangle + \langle Q_{trans} P, \eta \rangle + \langle Q_{ext} P, \eta \rangle, \quad \forall \eta \in \mathcal{V}(\mathbb{R}^N), \end{aligned}$$

with

$$\begin{aligned}
\langle Q_{int}P,\eta\rangle &:= -\gamma_1\int_\omega (\mathrm{div}(T(U_0+\nabla P)))\,\eta,\\
\langle Q_{trans}P,\eta\rangle &:= \int_{\partial\omega}\left[\gamma_1 T(U_0+(\nabla P)_{int})-\gamma_0 T(U_0+(\nabla P)_{ext})\right].n_{out}\;\eta,\\
\langle Q_{ext}P,\eta\rangle &:= -\gamma_0\int_{\mathbb{R}^N\setminus\omega}(\mathrm{div}(T(U_0+\nabla P)))\,\eta.
\end{aligned}$$

Hence P is a supersolution of Q in the half-space $\mathbb{R}^N_+$, that is

$$\langle QP,\eta\rangle\geq 0,\quad \forall\eta\in\mathcal{V}(\mathbb{R}^N),\mathrm{support}(\eta)\subset\mathbb{R}^N_+,\eta\geq 0\text{ a.e. },$$

if and only if the three following conditions are satisfied:

$$-\,\mathrm{div}(T(U_0+\nabla P))\geq 0,\qquad \forall x\in\omega, U_0.\ x>0; \tag{4.7.20}$$

$$\left[\gamma_1 T(U_0+(\nabla P)_{int}(x))-\gamma_0 T(U_0+(\nabla P)_{ext}(x))\right].n_{out}\geq 0,\qquad \forall x\in\partial\omega, U_0.\ x>0; \tag{4.7.21}$$

$$-\,\mathrm{div}(T(U_0+\nabla P))\geq 0,\qquad \forall x\in\mathbb{R}^N\setminus\bar\omega, U_0.\ x>0. \tag{4.7.22}$$

As we assume that $W=W_a$ for a given $a>0$, let us denote for simplicity

$$T_a(\varphi)=\sigma(|\varphi|^2)\ \varphi,\quad \forall\varphi\in\mathbb{R}^N,$$

with

$$\sigma(\lambda)=\left(a^2+\lambda\right)^{\frac{p-2}{2}},\quad \forall\lambda\in\mathbb{R}_+.$$

In ω and in $\mathbb{R}^N\setminus\bar\omega$, it holds

$$\begin{aligned}
\mathrm{div}(T_a(U_0+\nabla P)) &= \sigma(|U_0+\nabla P|^2)\,\Delta P+2\sigma'(|U_0+\nabla P|^2)\ D^2P(U_0+\nabla P,U_0+\nabla P)\\
&= \sigma(|U_0+\nabla P|^2)\left[\Delta P+\frac{p-2}{a^2+|U_0+\nabla P|^2}\ D^2P(U_0+\nabla P,U_0+\nabla P)\right].
\end{aligned} \tag{4.7.23}$$

Thus, as $\sigma>0$, the study of the sign in internal (resp. external) condition (4.7.20) (resp. (4.7.22)) can be carried out studying the sign of the term

$$\left(a^2+|U_0+\nabla P|^2\right)\Delta P+(p-2)\ D^2P(U_0+\nabla P,U_0+\nabla P). \tag{4.7.24}$$

1. It is obvious that $\mathrm{div}(T_a(U_0+\nabla P))=0$ in ω. Thus internal condition (4.7.20) is satisfied.
2. We now study external condition (4.7.22). For all $x\in\mathbb{R}^N$, $|x|>1$ and all $\varphi\in\mathbb{R}^N$, denoting

   $$r:=|x|\quad\text{and } e_r:=\frac{x}{|x|},$$

   an easy calculation shows that

   $$\nabla P(x)=kr^{-\beta}\left[U_0-\beta\,(U_0.e_r)\,e_r\right] \tag{4.7.25}$$

and

$$D^2P(x)(\varphi,\varphi) = \\ k\beta r^{-2-\beta}\left[(\beta+2)(U_0.x)\,(e_r.\varphi)^2 - 2(x.\varphi)(U_0.\varphi) - (U_0.x)\,|\varphi|^2\right]. \quad (4.7.26)$$

In particular

$$\Delta P(x) \;=\; -k\beta r^{-2-\beta}(U_0.x)\,[N-\beta]\,. \quad (4.7.27)$$

As β will be chosen such that $\beta < N$, it follows that $\Delta P < 0$ in the half-space $\{U_0.\,x > 0\}$. As we shall see, ΔP is going to provide the negative sign we are looking for in the whole term

$$\left(a^2 + |U_0 + \nabla P|^2\right)\Delta P + (p-2)\;D^2P(U_0+\nabla P, U_0+\nabla P).$$

In other words, we are looking for $\beta \in (N/2, N)$ such that the term

$$(p-2)\;D^2P(U_0+\nabla P, U_0+\nabla P)$$

is nowhere positive enough to invert the negative sign provided by the term

$$\left(a^2 + |U_0 + \nabla P|^2\right)\Delta P.$$

Let a given $x \in \mathbb{R}^N, |x| > 1$ and $U_0.\,x \geq 0$.
If $D^2P(x)(U_0+\nabla P(x), U_0+\nabla P(x)) \leq 0$, it follows immediately that

$$\left(a^2 + |U_0 + \nabla P(x)|^2\right)\Delta P(x) + (p-2)\;D^2P(x)(U_0+\nabla P(x), U_0+\nabla P(x)) \leq 0$$

and that external condition (4.7.22) is satisfied at point x.
Hence one can assume that $D^2P(x)(U_0+\nabla P(x), U_0+\nabla P(x)) > 0$.
Denoting

$$\tilde{\varphi} := U_0 + \nabla P(x) \quad \text{and} \quad \cos\theta := \frac{x}{|x|}.\frac{U_0}{|U_0|} = e_r.\frac{U_0}{|U_0|}$$

one obtains

$$\begin{aligned}
e_r.\tilde{\varphi} &= |U_0|\cos\theta\left[1 + kr^{-\beta}(1-\beta)\right], \\
x.\tilde{\varphi} &= (U_0.x)\left[1 + kr^{-\beta}(1-\beta)\right], \\
U_0.\tilde{\varphi} &= |U_0|^2\left[1 + kr^{-\beta}(1-\beta\cos^2\theta)\right], \\
|\tilde{\varphi}|^2 &= |U_0|^2\left[\sin^2\theta\left(1+kr^{-\beta}\right)^2 + \cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\right].
\end{aligned}$$

Thus formula (4.7.26) entails

$$D^2P(x)(\tilde{\varphi},\tilde{\varphi}) = k\beta r^{-2-\beta}(U_0.x)\,|U_0|^2\;\,f(r,\theta,k,\beta) \quad (4.7.28)$$

denoting

$$\begin{aligned}
f(r,\theta,k,\beta) \;:=\; &(\beta+1)\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2 \\
&-2\left(1+kr^{-\beta}(1-\beta)\right)\left(1+kr^{-\beta}(1-\beta\cos^2\theta)\right) \\
&-\sin^2\theta\left(1+kr^{-\beta}\right)^2.
\end{aligned}$$

In addition formula (4.7.27) yields

$$\begin{aligned}&\left(a^2+|\tilde{\varphi}|^2\right)\Delta P(x)\\&=k\beta r^{-2-\beta}(U_0.x)\,|U_0|^2\,[\beta-N]\left[\frac{a^2}{|U_0|^2}+\sin^2\theta\left(1+kr^{-\beta}\right)^2+\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\right].\end{aligned}\tag{4.7.29}$$

Hence the sign of (4.7.24), that is

$$\left(a^2+|\tilde{\varphi}|^2\right)\Delta P(x)+(p-2)\ D^2P(x)(\tilde{\varphi},\tilde{\varphi}),$$

is negative if and only if the sign of

$$[\beta-N]\left[\frac{a^2}{|U_0|^2}+\sin^2\theta\left(1+kr^{-\beta}\right)^2+\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\right]+(p-2)\ f(r,\theta,k,\beta)\tag{4.7.30}$$

is negative.
As $p\geq 2$, assuming that $\beta<N$, it thus suffices that

$$\begin{gathered}[\beta-N]\left[\frac{a^2}{|U_0|^2}+\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\right]\ +\\(p-2)\left[(\beta+1)\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2-2\left(1+kr^{-\beta}(1-\beta)\right)\left(1+kr^{-\beta}(1-\beta\cos^2\theta)\right)\right]\end{gathered}$$

be negative.
As after inequality (4.7.18), it holds

$$1+kr^{-\beta}(1-\beta\cos^2\theta)\geq 1+kr^{-\beta}(1-\beta)\geq 1+k(1-\beta)>0.$$

It follows

$$\begin{aligned}&-2\left(1+kr^{-\beta}(1-\beta)\right)\left(1+kr^{-\beta}(1-\beta\cos^2\theta)\right)\\&\qquad\leq -2\left(1+kr^{-\beta}(1-\beta)\right)^2\leq -2\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2.\end{aligned}$$

Hence it suffices that

$$[\beta-N]\,\frac{a^2}{|U_0|^2}+\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2[\beta-N+(p-2)(\beta+1-2)]\leq 0.\tag{4.7.31}$$

Assuming that $\beta>N/2\geq 1$, it follows from inequality (4.7.18) that

$$\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\leq 1,$$

Thus it suffices that

$$[\beta-N]\,\frac{a^2}{|U_0|^2}+[\beta-N+(p-2)(\beta-1)]\leq 0,$$

which is equivalent to

$$\beta \leq \frac{N\left(1+\frac{a^2}{|U_0|^2}\right)+(p-2)}{1+\frac{a^2}{|U_0|^2}+(p-2)}. \tag{4.7.32}$$

There exists $\beta \in (N/2, N)$ satisfying inequality (4.7.32) as soon as

$$\frac{N}{2} < \frac{N\left(1+\frac{a^2}{|U_0|^2}\right)+(p-2)}{1+\frac{a^2}{|U_0|^2}+(p-2)}. \tag{4.7.33}$$

The latter condition (4.7.33) is equivalent to

$$p < 2 + \left(1+\frac{a^2}{|U_0|^2}\right)\frac{N}{N-2} = \bar{p}$$

with the convention that $\bar{p} = +\infty$ if $N = 2$.
Therefore one concludes that for all $p \in [2, \bar{p})$, there exists $\beta \in (N/2, N)$, for instance choose

$$\beta := \frac{1}{2}\left(\frac{N}{2} + \frac{N\left(1+\frac{a^2}{|U_0|^2}\right)+(p-2)}{1+\frac{a^2}{|U_0|^2}+(p-2)}\right), \tag{4.7.34}$$

such that the external condition (4.7.22) is satisfied by function P.

3. Lastly let us prove that, for $\beta \in (N/2, N)$ defined by (4.7.34) and k defined by (4.4.14), function P satisfies the transmission condition (4.7.22), that is

$$\left[\gamma_1 T(U_0 + (\nabla P)_{int}(x)) - \gamma_0 T(U_0 + (\nabla P)_{ext}(x))\right].x \geq 0,\ \forall x, |x| = 1, U_0.\, x > 0.$$

As
$$T(\varphi) = \sigma(|\varphi|^2)\varphi,\ \forall \varphi \in \mathbb{R}^N$$
and as σ is an increasing function, a sufficient condition can be split into three conditions as follows

$$\left[\gamma_1(U_0 + (\nabla P)_{int}(x)) - \gamma_0(U_0 + (\nabla P)_{ext}(x))\right].\, x = 0, \qquad \forall x, |x| = 1, \tag{4.7.35}$$
$$|U_0 + (\nabla P)_{int}(x)| \geq |U_0 + (\nabla P)_{ext}(x)|, \qquad \forall x, |x| = 1, \tag{4.7.36}$$
$$(U_0 + (\nabla P)_{ext}(x)).\, x \geq 0, \qquad \forall x, |x| = 1, U_0.x > 0. \tag{4.7.37}$$

(a) After the definition (4.7.16) of P, the first condition (4.7.35) reads

$$\gamma_1(1+k) = \gamma_0\left(1 + k(1-\beta)\right).$$

which exactly provides the value of k chosen in definition (4.4.14).

(b) After the definition (4.7.16) of P, the second condition (4.7.36) reads

$$(1+k)^2 \geq (1+k)^2 + k\beta\cos^2\theta\left(-2 + k(\beta-2)\right),\ \forall \theta \in [-\pi/2, +\pi/2].$$

This condition is satisfied due to inequality (4.7.19).

4. Regarding the latter condition (4.7.37), it holds

$$\begin{aligned} (U_0 + (\nabla P)_{ext}(x)))\,.x &= [U_0 + k\,(U_0 - \beta(U_0.x)x)]\,.x \\ &= (U_0.x)\,[1 + k(1-\beta)] \\ &\geq 0, \end{aligned}$$

due to inequality (4.7.18).

This completes the proof that there exist $\beta \in (N/2, N)$ such that the three conditions (4.7.20), (4.7.21) and (4.7.22) are satisfied. Hence the proof of Proposition 4.4.6 is completed.

Proof of proposition 4.4.8

By assumption it holds $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ and $W = W_a$ for some $a > 0$ and $p \in [2, \bar{p})$. Recall that by symmetry there exists an element $\tilde{H}$ of the class H such that

$$\tilde{H}(x) = 0, \quad \forall x \in \mathbb{R}^N, U_0.\ x = 0.$$

Let P the supersolution defined in Proposition (4.4.6). For all $\eta \in \mathcal{V}(\mathbb{R}^N)$ such that $support(\eta) \subset \mathbb{R}^N_+$ and $\eta \geq 0$ a.e., it holds

$$\langle QP, \eta \rangle \geq 0.$$

As by definition of H, it holds $QH = 0$, it follows that

$$\langle QP - QH, \eta \rangle \geq 0,$$

that is

$$\int_{\mathbb{R}^N} \gamma\,[T(U_0 + \nabla P) - T(U_0 + \nabla H)]\,.\nabla\eta \geq 0. \tag{4.7.38}$$

Denote χ the characteristic function of half-space $\mathbb{R}^N_+$, that is

$$\forall x \in \mathbb{R}^N, \begin{cases} \chi(x) = 1 & \text{if } x \in \mathbb{R}^N_+, \\ \chi(x) = 0 & \text{if } x \notin \mathbb{R}^N_+. \end{cases}$$

As $P = \tilde{H} = 0$ in the hyperplane $(\mathbb{R}U_0)^\perp$, the test function defined by

$$\eta(x) := \chi(x)\ \max(0, \tilde{H}(x) - P(x)), \quad \forall x \in \mathbb{R}^N$$

satisfies the conditions $\eta \in \mathcal{V}(\mathbb{R}^N)$, $support(\eta) \subset \mathbb{R}^N_+$ and $\eta \geq 0$ a.e. . Hence it can be plugged into inequality (4.7.38). It follows

$$\int_{\{\tilde{H}>P\}\ \cap\ \mathbb{R}^N_+} \gamma\,[T(U_0 + \nabla P) - T(U_0 + \nabla H)]\,.(\nabla H - \nabla P) \geq 0,$$

that is

$$\int_{\{\tilde{H}>P\}\ \cap\ \mathbb{R}^N_+} \gamma\,[T(U_0 + \nabla P) - T(U_0 + \nabla H)]\,.(\nabla P - \nabla H) \leq 0.$$

Moreover after the ellipticity condition (5), there exists $c > 0$ such that

$$c \int_{\{\tilde{H}>P\} \,\cap\, \mathbb{R}^N_+} |\nabla H - \nabla P|^p + |\nabla H - \nabla P|^2 \leq \int_{\{\tilde{H}>P\} \,\cap\, \mathbb{R}^N_+} \gamma \left[T(U_0 + \nabla P) - T(U_0 + \nabla H)\right].(\nabla P - \nabla H).$$

Hence

$$\int_{\{\tilde{H}>P\} \,\cap\, \mathbb{R}^N_+} \gamma \left[T(U_0 + \nabla P) - T(U_0 + \nabla H)\right].(\nabla P - \nabla H) \geq 0.$$

One concludes

$$\int_{\mathbb{R}^N} |\nabla \eta|^p + |\nabla \eta|^2 = \int_{\{\tilde{H}>P\} \,\cap\, \mathbb{R}^N_+} |\nabla H - \nabla P|^p + |\nabla H - \nabla P|^2 = 0$$

Due to the Poincaré inequality stated in Corollary (3.5.1) on page 49, it follows that $\eta = 0$ in $\mathcal{V}(\mathbb{R}^N)$. Hence $P \geq \tilde{H}$ a.e. in $\mathbb{R}^N_+$.

Similarly, one obtains from Lemma 4.4.7 that $\tilde{H} \geq 0$ a.e. in $\mathbb{R}^N_+$.

As H is an odd function with respect to the first coordinate x_1, i.e. along the line $\mathbb{R}U_0$, it follows immediately that for almost every $x \in \mathbb{R}^N$,

$$\begin{cases} 0 \leq \tilde{H}(x) \leq P(x), & \text{if } x \in \mathbb{R}^N_+, \\ -P(x) \leq \tilde{H}(x) \leq 0, & \text{if } x \notin \mathbb{R}^N_+. \end{cases}$$

As $P \in L^\infty(\mathbb{R}^N)$ and as

$$P(y) = O\left(|y|^{-\tau}\right) \quad \text{as} \quad |y| \to +\infty.$$

with $\tau := \beta - 1 > N/2 - 1$, one eventually concludes that $H \in L^\infty(\mathbb{R}^N)$ and that

$$\tilde{H}(y) = O\left(|y|^{-\tau}\right) \quad \text{as} \quad |y| \to +\infty.$$

Proof of Lemma 4.4.11

As $H \in \mathcal{V}(\mathbb{R}^N)$, by definition it holds $\nabla H \in L^2(\mathbb{R}^N)$. Moreover according to Assumption 4.4.10, $\tilde{H} \in L^\infty(\mathbb{R}^N)$. As in addition $w_2 \in L^2(\mathbb{R}^N)$, it follows that $w_2 \tilde{H} \in L^2(\mathbb{R}^N)$, which completes the proof of the assertion $H \in \mathcal{H}(\mathbb{R}^N)$.

Proof of Proposition 4.4.12

Let us begin proving a technical lemma. Recall $0 < \rho < R$ defined in (4.2.1) such that $\omega \subset\subset B(0, \rho) \subset B(0, R) \subset\subset \Omega$.

Let $\theta : \mathbb{R}^N \to \mathbb{R}$ a smooth function such that

$$\theta(x) = 0, \ \forall x \in \overline{B}(0, \rho) \quad \text{and} \quad \theta(x) = 1, \ \forall x \in \mathbb{R}^N \setminus B(0, R). \tag{4.7.39}$$

Recall

$$H_\varepsilon(x) := \varepsilon \tilde{H}(\varepsilon^{-1} x), \quad \forall x \in \Omega. \tag{4.7.40}$$

Let

$$\begin{aligned} \kappa_\varepsilon \ : \ & \Omega \to \mathbb{R} \\ & x \mapsto \theta(x) H_\varepsilon(x). \end{aligned}$$

Lemma 4.7.2. *It holds* $\kappa_\varepsilon \in W^{1,p}(\Omega)$ *and* $H_\varepsilon - \kappa_\varepsilon \in \mathcal{V}$. *Moreover*

$$\|\nabla \kappa_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla \kappa_\varepsilon\|^2_{L^2(\Omega)} \;=\; o(\varepsilon^N). \tag{4.7.41}$$

Proof. Denote $C_\theta := \max(\|\theta\|_{L^\infty(\mathbb{R}^N)}, \|\nabla\theta\|_{L^\infty(\mathbb{R}^N)})$.

– Since $|\kappa_\varepsilon(x)| \le C_\theta |H_\varepsilon(x)|$ for a.e. $x \in \Omega$, it follows from $H_\varepsilon \in L^p(\Omega)$ that $\kappa_\varepsilon \in L^p(\Omega)$.

– According to the Leibniz formula it holds

$$\nabla \kappa_\varepsilon(x) = \nabla\theta(x)\; H_\varepsilon(x) + \theta(x)\; \nabla H_\varepsilon(x) \quad \text{for a.e. } x \in \Omega.$$

Hence by convexity, for a.e. $x \in \Omega$

$$|\nabla \kappa_\varepsilon(x)|^p \le 2^{p-1} C_\theta^p \left(|H_\varepsilon(x)|^p + |\nabla H_\varepsilon(x)|^p\right).$$

Thus $H_\varepsilon \in W^{1,p}(\Omega)$ entails that $\nabla\kappa_\varepsilon \in L^p(\Omega)$.

One concludes $\kappa_\varepsilon \in W^{1,p}(\Omega)$.

Moreover by definition of θ, it holds $H_\varepsilon - \kappa_\varepsilon = 0$ on $\partial\Omega$. Thus according to the trace theorem in $W^{1,p}(\Omega)$, it follows $H_\varepsilon - \kappa_\varepsilon \in W^{1,p}_0(\Omega) = \mathcal{V}$.

Let us now prove estimate (4.7.41). Let $C := \max(2^{p-1}C_\theta^p, 2C_\theta^2)$. By convexity, for a.e. $x \in \Omega$ it holds

$$|\nabla\kappa_\varepsilon(x)|^p + |\nabla\kappa_\varepsilon(x)|^2 \le C\left(|H_\varepsilon(x)|^p + |H_\varepsilon(x)|^2 + |\nabla H_\varepsilon(x)|^p + |\nabla H_\varepsilon(x)|^2\right). \tag{4.7.42}$$

1. In $B(0,\rho)$, $\theta = 0$. Thus

$$\int_{B(0,\rho)} |\nabla\kappa_\varepsilon|^p + |\nabla\kappa_\varepsilon|^2 = 0. \tag{4.7.43}$$

2. Then let's integrate in $B(0,R)\setminus B(0,\rho)$. Making a change of scale and applying the asymptotic behavior of $\tilde{H}$ given by (4.4.18), one obtains

$$\begin{aligned}
\int_{B(0,R)\setminus B(0,\rho)} |H_\varepsilon|^p + |H_\varepsilon|^2 &= \varepsilon^N \int_{B(0,R/\varepsilon)\setminus B(0,\rho/\varepsilon)} \varepsilon^p \left|\tilde{H}\right|^p + \varepsilon^2\left|\tilde{H}\right|^2 \\
&\le \varepsilon^N\, O\left(\frac{R}{\varepsilon}\right)^N\, O\left(\varepsilon^p\left(\frac{\varepsilon}{\rho}\right)^{p\tau} + \varepsilon^2\left(\frac{\varepsilon}{\rho}\right)^{2\tau}\right) \\
&\le O\left(\varepsilon^{p(1+\tau)} + \varepsilon^{2(1+\tau)}\right) = o(\varepsilon^N),
\end{aligned}$$

since $p(1+\tau) \ge 2(1+\tau) > N$.

Recall $\nabla H \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Thus

$$\int_{B(0,R)\setminus B(0,\rho)} |\nabla H_\varepsilon|^p + |\nabla H_\varepsilon|^2 \;\le\; \varepsilon^N \int_{\mathbb{R}^N\setminus B(0,\rho/\varepsilon)} |\nabla H|^p + |\nabla H|^2 = o\left(\varepsilon^N\right).$$

Therefore integrating inequality (4.7.42) in $B(0,R)\setminus B(0,\rho)$ entails

$$\int_{B(0,R)\setminus B(0,\rho)} |\nabla\kappa_\varepsilon|^p + |\nabla\kappa_\varepsilon|^2 \;=\; o(\varepsilon^N). \tag{4.7.44}$$

3. Lastly it holds $\kappa_\varepsilon = H_\varepsilon$ in $\Omega\setminus\overline{B}(0,R)$ and thus $\nabla\kappa_\varepsilon = \nabla H_\varepsilon$ in $\Omega\setminus\overline{B}(0,R)$. Since $\nabla H \in L^p(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$, it follows

$$\int_{\Omega\setminus B(0,R)} |\nabla\kappa_\varepsilon|^p + |\nabla\kappa_\varepsilon|^2 \le \varepsilon^N \int_{\mathbb{R}^N\setminus B(0,R/\varepsilon)} |\nabla H|^p + |\nabla H|^2 = o\left(\varepsilon^N\right). \tag{4.7.45}$$

Gathering (4.7.43), (4.7.44) and (4.7.45), one eventually obtains

$$\|\nabla\kappa_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla\kappa_\varepsilon\|^2_{L^2(\Omega)} = o(\varepsilon^N).$$

□

We now prove Proposition 4.4.12.

1. First we prove inequality (4.4.21), that is

$$\|\nabla h_\varepsilon - \nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^2_{L^2(\Omega)} = o(\varepsilon^N).$$

For all $\eta \in \mathcal{V}$, define $\eta_\varepsilon \in \mathcal{V}(\mathbb{R}^N)$ by $\eta_\varepsilon(y) := \varepsilon^{-1}\eta(\varepsilon y)$ for all $y \in \Omega/\varepsilon$ and $\eta_\varepsilon(y) := 0$ for all $y \in \mathbb{R}^N \setminus (\Omega/\varepsilon)$. Applying variational formulation (4.4.5) to η_ε and making the change of scale backward, one obtains

$$\int_\Omega \gamma_\varepsilon \left[T(U_0 + \nabla H_\varepsilon) - T(U_0)\right].\nabla\eta = -(\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(U_0).\nabla\eta. \tag{4.7.46}$$

Calculating the difference with variational form (4.4.4) yields

$$\int_\Omega \gamma_\varepsilon \left[T(U_0 + \nabla h_\varepsilon) - T(U_0 + \nabla H_\varepsilon)\right].\nabla\eta = 0, \quad \forall\, \eta \in \mathcal{V}. \tag{4.7.47}$$

Recall function κ_ε studied in Lemma 4.7.2. It holds $H_\varepsilon - \kappa_\varepsilon \in \mathcal{V}$. Plugging $\eta = h_\varepsilon - (H_\varepsilon - \kappa_\varepsilon) \in \mathcal{V}$ in (4.7.47) one obtains

$$\begin{aligned}\int_\Omega \gamma_\varepsilon &\left[T(U_0 + \nabla h_\varepsilon) - T(U_0 + \nabla H_\varepsilon)\right].(\nabla h_\varepsilon - \nabla H_\varepsilon) \\ &= -\int_\Omega \gamma_\varepsilon \left[T(U_0 + \nabla h_\varepsilon) - T(U_0 + \nabla H_\varepsilon)\right].\nabla\kappa_\varepsilon.\end{aligned} \tag{4.7.48}$$

Looking at the term on the left hand side of (4.7.48) it follows from condition (5) that

$$\begin{aligned}\underline{\gamma} c\, &\left(\|\nabla h_\varepsilon - \nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^2_{L^2(\Omega)}\right) \\ &\leq \int_\Omega \gamma_\varepsilon \left[T(U_0 + \nabla h_\varepsilon) - T(U_0 + \nabla H_\varepsilon)\right].(\nabla h_\varepsilon - \nabla H_\varepsilon).\end{aligned} \tag{4.7.49}$$

Looking at the term on the right hand side of (4.7.48) and applying inequality (4.1.1) with $M := |U_0| + \|\nabla H\|_{L^\infty(\mathbb{R}^N)}$, one obtains

$$\begin{aligned}&\left|\int_\Omega \left[T(U_0 + \nabla h_\varepsilon) - T(U_0 + \nabla H_\varepsilon)\right].\nabla\kappa_\varepsilon\right| \\ &\leq \int_\Omega \left[b_1 |\nabla h_\varepsilon - \nabla H_\varepsilon| + b_{p-1} |\nabla h_\varepsilon - \nabla H_\varepsilon|^{p-1}\right].|\nabla\kappa_\varepsilon| \\ &\leq b_1 \|\nabla h_\varepsilon - \nabla H_\varepsilon\|_{L^2(\Omega)} \|\nabla\kappa_\varepsilon\|_{L^2(\Omega)} + b_{p-1} \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^{\frac{p}{q}}_{L^p(\Omega)} \|\nabla\kappa_\varepsilon\|_{L^p(\Omega)}.\end{aligned} \tag{4.7.50}$$

Gathering (4.7.48), (4.7.49) and (4.7.50) as well as estimates (4.7.12) and (4.7.41), it follows that

$$\begin{aligned}
&\underline{\gamma} c \ \left(\|\nabla h_\varepsilon - \nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^2_{L^2(\Omega)}\right) \\
&\le \overline{\gamma} b_1 \left(O(\varepsilon^N)\right)^{\frac{1}{2}} \left(o(\varepsilon^N)\right)^{\frac{1}{2}} + \overline{\gamma} b_{p-1} \left(O(\varepsilon^N)\right)^{\frac{1}{q}} \left(o(\varepsilon^N)\right)^{\frac{1}{p}} \\
&\qquad\qquad = O(\varepsilon^{\frac{N}{2}}) \ o(\varepsilon^{\frac{N}{2}}) + O(\varepsilon^{\frac{N}{q}}) \ o(\varepsilon^{\frac{N}{p}}) = o(\varepsilon^N)
\end{aligned}$$

which is the claimed estimate (4.4.21).

2. We move on to the proof of inequality (4.4.22). Let $\alpha > 0$ and $r \in (0,1)$. Since $\nabla H \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ and $r - 1 < 0$ it holds
$$\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla H_\varepsilon|^p + |\nabla H_\varepsilon|^2 \ \le \ \varepsilon^N \int_{\mathbb{R}^N\setminus B(0,\alpha\varepsilon^{r-1})} |\nabla H|^p + |\nabla H|^2 = o(\varepsilon^N).$$
The latter estimate combined with estimate (4.4.21) entails by convexity that
$$\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2 = o(\varepsilon^N)$$
which is the claimed estimate (4.4.22).

3. We now prove estimate (4.4.23). After Lemma 4.4.1, ∇u_0 is β- Hölder continuous at point $x_0 = 0$ for some $\beta > 0$. Hence there exist $\delta > 0$ and $L > 0$ such that
$$|\nabla u_0(x) - U_0| \le L\,|x|^\beta, \quad \forall x \in B(0,\delta).$$
To apply estimate (4.4.22), we choose $\alpha := 1$ and $r = 1/2$. For all $\varepsilon \in (0, \delta^2)$, according to estimates (4.4.10) and (4.4.22) it follows
$$\begin{aligned}
&\int_\Omega |\nabla u_0 - U_0| \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) \\
&\le \int_{B(0,\alpha\varepsilon^r)} L\,|x|^\beta \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) + 2\,\|\nabla u_0\|_{L^\infty(\Omega)} \int_{\Omega\setminus B(0,\alpha\varepsilon^r)} \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) \\
&\qquad\qquad \le L\alpha^\beta \varepsilon^{r\beta} \ O(\varepsilon^N) + o(\varepsilon^N) = o(\varepsilon^N),
\end{aligned}$$
which completes the proof of estimate (4.4.23).

4. For all $p \in (4,\infty)$ and for all $\lambda \in \mathbb{R}_+$ it holds $\lambda^{p-2} \le \lambda^2 + \lambda^p$. Hence the claimed estimate (4.4.24)
$$\forall\ p \in (4,\infty), \int_\Omega |\nabla u_0 - U_0|\,|\nabla h_\varepsilon|^{p-2} = o(\varepsilon^N),$$
follows immediately from estimate (4.4.23).

5. Similarly, for all $p \in (3,\infty)$ and for all $\lambda \in \mathbb{R}_+$ it holds $\lambda^{p-1} \le \lambda^2 + \lambda^p$. Hence the claimed estimate (4.4.25)
$$\forall\ p \in (3,\infty), \int_\Omega |\nabla u_0 - U_0|\,|\nabla h_\varepsilon|^{p-1} = o(\varepsilon^N),$$
follows immediately from estimate (4.4.23).

6. Regarding estimate (4.4.26), Cauchy-Schwarz's inequality and estimates (4.4.21), (4.4.10) and (4.4.11) entail that

$$\begin{aligned}&\int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right) \\ &\leq \|\nabla h_\varepsilon - \nabla H_\varepsilon\|_{L^2(\Omega)} \left[\|\nabla h_\varepsilon\|_{L^2(\Omega)} + \|\nabla H_\varepsilon\|_{L^2(\Omega)}\right] = o(\varepsilon^{\frac{N}{2}})\, O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^N)\end{aligned}$$

which completes the proof of estimate (4.4.26).

7. Lastly let $p \in (3, \infty)$. For all $\lambda \in \mathbb{R}_+$ it holds $\lambda^{p-2} \leq \lambda + \lambda^{p-1}$. Hence due to estimates (4.4.21), (4.4.10), (4.4.11), (4.4.26) and to Hölder's inequality one obtains

$$\begin{aligned}&\int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right)^{p-2} \\ &\leq \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right) + \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right)^{p-1} \\ &\leq o(\varepsilon^N) + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|_{L^p(\Omega)} \||\nabla h_\varepsilon| + |\nabla H_\varepsilon|\|^{\frac{p}{q}}_{L^p(\Omega)} \\ &\qquad\qquad = o(\varepsilon^N) + o(\varepsilon^{\frac{N}{p}})\; O(\varepsilon^{\frac{N}{q}}) = o(\varepsilon^N)\end{aligned}$$

which is the claimed estimate (4.4.27).

Proof of Proposition 4.4.13

1. First let's prove inequality (4.4.28), that is

$$\|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|^2_{L^2(\Omega)} \quad = \quad o(e^N).$$

For all $\eta \in \mathcal{V}$, calculating the difference between variational forms (4.4.3) and (4.4.4) yields

$$\begin{aligned}&\int_\Omega \gamma_\varepsilon \left[T(\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T(U_0 + \nabla h_\varepsilon) + T(U_0) - T(\nabla u_0)\right].\nabla\eta \\ &\qquad\qquad = -(\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} (T(\nabla u_0) - T(U_0)).\nabla\eta.\end{aligned}$$

That is

$$\begin{aligned}&\int_\Omega \gamma_\varepsilon \left[T(\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T(\nabla u_0 + \nabla h_\varepsilon)\right].\nabla\eta \\ &= -(\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} (T(\nabla u_0) - T(U_0)).\nabla\eta + \int_\Omega \gamma_\varepsilon \left[T(\nabla u_0) - T(U_0)\right].\nabla\eta \\ &\qquad\qquad + \int_\Omega \gamma_\varepsilon \left[T(U_0 + \nabla h_\varepsilon) - T(\nabla u_0 + \nabla h_\varepsilon)\right].\nabla\eta.\end{aligned}$$

For all $\alpha > 0$ and for all $r \in (0,1)$, splitting the two latter integrals into $B(0, \alpha\varepsilon^r)$

and $\Omega \setminus B(0,\alpha\varepsilon^r)$, one may rewrite the latter equality as follows

$$\begin{aligned}
&\int_\Omega \gamma_\varepsilon \left[T(\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T(\nabla u_0 + \nabla h_\varepsilon)\right].\nabla\eta = \\
&-(\gamma_1-\gamma_0)\int_{\omega_\varepsilon}(T(\nabla u_0) - T(U_0)).\nabla\eta \\
&+\int_{B(0,\alpha\varepsilon^r)}\gamma_\varepsilon\left[T(\nabla u_0) - T(U_0)\right].\nabla\eta \\
&+\int_{B(0,\alpha\varepsilon^r)}\gamma_\varepsilon\left[T(U_0+\nabla h_\varepsilon) - T(\nabla u_0+\nabla h_\varepsilon)\right].\nabla\eta \\
&+\int_{\Omega\setminus B(0,\alpha\varepsilon^r)}\gamma_\varepsilon\left[T(U_0+\nabla h_\varepsilon) - T(U_0)\right].\nabla\eta \\
&\qquad\qquad+\int_{\Omega\setminus B(0,\alpha\varepsilon^r)}\gamma_\varepsilon\left[T(\nabla u_0) - T(\nabla u_0+\nabla h_\varepsilon)\right].\nabla\eta.
\end{aligned}$$

Plugging the test function $\eta = \tilde{u}_\varepsilon - h_\varepsilon \in \mathcal{V}$ and applying condition (5) it follows that

$$c\underline{\gamma}\left(\|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon\|^2_{L^2(\Omega)}\right) \leq \sum_{i=1}^5 \mathcal{E}_i(\varepsilon),$$

with

$$\mathcal{E}_1(\varepsilon) = -(\gamma_1-\gamma_0)\int_{\omega_\varepsilon}(T(\nabla u_0) - T(U_0)).(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon), \tag{4.7.51}$$

$$\mathcal{E}_2(\varepsilon) = \int_{B(0,\alpha\varepsilon^r)}\gamma_\varepsilon\left[T(\nabla u_0) - T(U_0)\right].(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon), \tag{4.7.52}$$

$$\mathcal{E}_3(\varepsilon) = \int_{B(0,\alpha\varepsilon^r)}\gamma_\varepsilon\left[T(U_0+\nabla h_\varepsilon) - T(\nabla u_0+\nabla h_\varepsilon)\right].(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon), \tag{4.7.53}$$

$$\mathcal{E}_4(\varepsilon) = \int_{\Omega\setminus B(0,\alpha\varepsilon^r)}\gamma_\varepsilon\left[T(U_0+\nabla h_\varepsilon) - T(U_0)\right].(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon), \tag{4.7.54}$$

$$\mathcal{E}_5(\varepsilon) = \int_{\Omega\setminus B(0,\alpha\varepsilon^r)}\gamma_\varepsilon\left[T(\nabla u_0) - T(\nabla u_0+\nabla h_\varepsilon)\right].(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon). \tag{4.7.55}$$

Hence it suffices to prove that there exist $\alpha > 0$ and $r \in (0,1)$ such that

$$\mathcal{E}_i(\varepsilon) = o(\varepsilon^N), \quad \forall i, 1 \leq i \leq 5.$$

To apply estimate (4.4.22), we choose $\alpha := \rho$ (see (4.2.1)) and

$$r := \frac{1}{2}\left(\frac{N}{2\beta+N} + 1\right) \in (0,1).$$

In particular after (4.2.1), it holds $\omega \subset\subset B(0,\rho) = B(0,\alpha)$.
After Lemma 4.4.1 there exists $\beta > 0$ such that ∇u_0 is β-Hölder continuous at point $x_0 = 0$. In addition, after condition (1), T is Lipschitz-continuous at point U_0. Hence there exist $\delta > 0$ and $L > 0$ such that

$$\max(|\nabla u_0(x) - U_0|, |T(\nabla u_0(x)) - T(U_0)|) \leq L|x|^\beta \quad \forall x \in \Omega, |x| \leq \delta. \tag{4.7.56}$$

In addition for all $\varepsilon \in \left(0, \min\left(1, \left(\frac{\delta}{\rho}\right)^{\frac{1}{r}}\right)\right)$ it holds

$$\omega_\varepsilon \subset B(0,\rho\varepsilon) \subset B(0,\rho\varepsilon^r) \subset B(0,\delta).$$

Then one may estimate $\mathcal{E}_i(\varepsilon)$ for $1 \leq i \leq 5$ as follows.

(a) Applying Cauchy-Schwarz's inequality, it follows from estimates (4.7.56) and (4.7.11) that

$$\begin{aligned}|\mathcal{E}_1(\varepsilon)| &\leq 2\overline{\gamma}\int_{\omega_\varepsilon} |(T(\nabla u_0) - T(U_0))|\,|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon| \\ &\leq 2\overline{\gamma}L\rho^\beta\varepsilon^\beta\ \ |\omega|\,\varepsilon^{\frac{N}{2}}\ O\left(\varepsilon^{\frac{N}{2}}\right) = o(\varepsilon^N).\end{aligned}$$

(b) Similarly after estimate (4.7.11) and (4.7.56) and Cauchy-Schwarz's inequality, it holds

$$\begin{aligned}|\mathcal{E}_2(\varepsilon)| &= \left|\int_{B(0,\rho\varepsilon^r)} \gamma_\varepsilon\left[T(\nabla u_0) - T(U_0)\right].(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon)\right| \\ &\leq \overline{\gamma}L\rho^\beta\varepsilon^{\beta r}\ O\left(\varepsilon^{\frac{rN}{2}}\right)\ O\left(\varepsilon^{\frac{N}{2}}\right) \\ &= O\left(\varepsilon^{\frac{r(2\beta+N)+N}{2}}\right) = o(\varepsilon^N).\end{aligned}$$

(c) After condition (6), it holds

$$\begin{aligned}&|\mathcal{E}_3(\varepsilon)| = \left|\int_{B(0,\rho\varepsilon^r)} \gamma_\varepsilon\left[T(U_0 + \nabla h_\varepsilon) - T(\nabla u_0 + \nabla h_\varepsilon)\right].(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon)\right| \\ &\leq \overline{\gamma}C\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0|\left[1 + |\nabla u_0 + \nabla h_\varepsilon|^{p-2} + |U_0 - \nabla u_0|^{p-2}\right]|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon|\,.\end{aligned} \tag{4.7.57}$$

Let us estimate the upper bound of (4.7.57).

– First after estimates (4.7.11) and (4.7.56) and Cauchy-Schwarz's inequality it holds

$$\begin{aligned}&\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0|\left[1 + |U_0 - \nabla u_0|^{p-2}\right]|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon| \\ &\leq L\rho^\beta\varepsilon^{\beta r}\left[1 + \left(L\delta^\beta\right)^{p-2}\right]\int_{B(0,\rho\varepsilon^r)} |\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon| \\ &\qquad\qquad \leq \varepsilon^{\beta r}O\left(\varepsilon^{\frac{rN}{2}}\right)\ O\left(\varepsilon^{\frac{N}{2}}\right) = O\left(\varepsilon^{\frac{r(2\beta+N)+N}{2}}\right) = o(\varepsilon^N).\end{aligned}$$

– Then since $p \geq 2$,

$$\begin{aligned}&\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0|\,|\nabla u_0 + \nabla h_\varepsilon|^{p-2}\,|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon| \\ &\quad\leq 2^{p-2}\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0|\left(|\nabla u_0|^{p-2} + |\nabla h_\varepsilon|^{p-2}\right)|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon|\,.\end{aligned}$$

– On one hand after estimates (4.7.11) and (4.7.56) and Cauchy-Schwarz's inequality, it holds

$$\begin{aligned}&\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0|\,|\nabla u_0|^{p-2}\,|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon| \\ &\leq L\rho^\beta\varepsilon^{\beta r}\left(|U_0| + L\delta^\beta\right)^{p-2}O\left(\varepsilon^{\frac{rN}{2}}\right)\ O\left(\varepsilon^{\frac{N}{2}}\right) \\ &\qquad\qquad = O\left(\varepsilon^{\frac{r(2\beta+N)+N}{2}}\right) = o(\varepsilon^N).\end{aligned}$$

– On the other hand, according to (4.7.56), (4.4.10) and (4.7.11) and Cauchy-Schwarz's inequality

$$\begin{aligned}
&\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0| \, |\nabla h_\varepsilon|^{p-2} \, |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \\
&\leq L\rho^\beta \varepsilon^{\beta r} \left(\int_{B(0,\rho\varepsilon^r)} |\nabla h_\varepsilon|^p \right)^{\frac{p-2}{2}} \left(\int_{B(0,\rho\varepsilon^r)} |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon|^{\frac{p}{2}} \right)^{\frac{2}{p}} \\
&\qquad \leq L\rho^\beta \varepsilon^{\beta r} \left(O(\varepsilon^N) \right)^{\frac{p-2}{2}} \left[O(\varepsilon^{\frac{rN}{2}}) O(\varepsilon^{\frac{N}{2}}) \right]^{\frac{2}{p}} = O\left(\varepsilon^s\right) = o(\varepsilon^N).
\end{aligned}$$

since

$$\begin{aligned}
s &:= \beta r + \frac{N(p-2)}{2} + \frac{rN}{p} + \frac{N}{p} = \frac{r(p\beta + N)}{p} + \frac{N(p-2)}{2} + \frac{N}{p} \\
&\geq \frac{r(2\beta + N)}{p} + \frac{N(p-2)}{2} + \frac{N}{p} \\
&\qquad > \frac{2N}{p} + \frac{N(p-2)}{2} = N\left(1 + \frac{(p-2)^2}{2p}\right) \geq N.
\end{aligned}$$

Hence the upper bound of (4.7.57) is $o(\varepsilon^N)$. Therefore one concludes from (4.7.57) that

$$\mathcal{E}_3(\varepsilon) \;=\; o(\varepsilon^N).$$

(d) After inequality (4.1.1) it holds

$$\begin{aligned}
|\mathcal{E}_4(\varepsilon)| &= \left| \int_{\Omega\setminus B(0,\rho\varepsilon^r)} \gamma_\varepsilon \left[T(U_0 + \nabla h_\varepsilon) - T(U_0) \right] . (\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon) \right| \\
&\leq \overline{\gamma} \int_{\Omega\setminus B(0,\rho\varepsilon^r)} \left[b_1 \, |\nabla h_\varepsilon| + b_{p-1} \, |\nabla h_\varepsilon|^{p-1} \right] |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \, .
\end{aligned}$$

Applying Hölder's inequality and estimates (4.7.11) and (4.4.22), it follows

$$\begin{aligned}
|\mathcal{E}_4(\varepsilon)| &\leq \overline{\gamma} b_1 \left(\int_{\Omega\setminus B(0,\rho\varepsilon^r)} |\nabla h_\varepsilon|^2 \right)^{\frac{1}{2}} \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^2(\Omega)} \\
&\quad + \overline{\gamma} b_{p-1} \left(\int_{\Omega\setminus B(0,\rho\varepsilon^r)} |\nabla h_\varepsilon|^p \right)^{\frac{1}{q}} \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^p(\Omega)} \\
&\leq o(\varepsilon^{\frac{N}{2}}) \; O(\varepsilon^{\frac{N}{2}}) + o(\varepsilon^{\frac{N}{q}}) \; O(\varepsilon^{\frac{N}{p}}) = o(\varepsilon^N).
\end{aligned}$$

(e) Lastly

$$\mathcal{E}_5(\varepsilon) \;=\; \int_{\Omega\setminus B(0,\rho\varepsilon^r)} \gamma_\varepsilon \left[T(\nabla u_0) - T(\nabla u_0 + \nabla h_\varepsilon) \right] . (\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon).$$

After Lemma 4.4.1 it holds $\nabla u_0 \in L^\infty(\Omega)$. Thus one can apply again inequality (4.1.1) and proves exactly as for $\mathcal{E}_4(\varepsilon)$ that $\mathcal{E}_5(\varepsilon) = o(\varepsilon^N)$.

Therefore we conclude that $\|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|^2_{L^2(\Omega)} = o(e^N)$.

2. Regarding estimate (4.4.29), Cauchy-Schwarz's inequality and estimates (4.4.28), (4.4.9) and (4.4.10) entail that

$$\begin{aligned}
&\int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon| \right) \\
&\leq \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^2(\Omega)} \left(\|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)} + \|\nabla h_\varepsilon\|_{L^2(\Omega)} \right) \\
&\qquad\qquad = o(\varepsilon^{\frac{N}{2}}) \; O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^N)
\end{aligned}$$

which completes the proof of estimate (4.4.29).

3. Let $p \in (3, \infty)$. For all $\lambda \in \mathbb{R}_+$ it holds $\lambda^{p-2} \leq \lambda + \lambda^{p-1}$. Hence due to estimates (4.4.29), (4.4.28), (4.4.9), (4.4.10) and to Hölder's inequality one obtains

$$\begin{aligned}
&\int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right)^{p-2} \\
&\leq \int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right) + \int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right)^{p-1} \\
&\leq o(\varepsilon^N) + \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^p(\Omega)} \, \||\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\|_{L^p(\Omega)}^{\frac{p}{q}} \\
&\qquad\qquad\qquad\qquad = o(\varepsilon^N) + o(\varepsilon^{\frac{N}{p}}) \; O(\varepsilon^{\frac{N}{q}}) = o(\varepsilon^N)
\end{aligned}$$

which is the claimed estimate (4.4.30).

4. Eventually estimate (4.4.31) immediately follows by convexity from estimates (4.4.22) and (4.4.28).

4.7.3 Proofs about the variation of the adjoint state

Proof of Proposition 4.5.3

1. We first prove estimate (4.5.8). After Lemma 4.4.1, it holds $u_0 \in C^{1,\beta}(\overline{\Omega})$. Hence $\nabla u_0 \in L^\infty(\overline{\Omega})$ and then $DT(\nabla u_0) \in L^\infty(\Omega)$. Moreover we assumed $\nabla v_0 \in L^\infty(\Omega)$. Due to the ellipticity of DT stated in condition (4), applying test function $\eta = \tilde{v}_\varepsilon$ in the variational form (4.5.4) defining $\tilde{v}_\varepsilon$, it holds:

$$\begin{aligned}
\underline{\gamma}\; c \int_\Omega |\nabla \tilde{v}_\varepsilon|^2 \;&\leq\; \int_\Omega \gamma_\varepsilon DT(\nabla u_0)(\nabla \tilde{v}_\varepsilon, \nabla \tilde{v}_\varepsilon) \\
&=\; -(\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} DT(\nabla u_0) \nabla v_0 . \nabla \tilde{v}_\varepsilon \\
&\leq\; |\gamma_1 - \gamma_0| \, \|DT(\nabla u_0)\|_{L^\infty} \|\nabla v_0\|_{L^\infty} \int_{\omega_\varepsilon} |\nabla \tilde{v}_\varepsilon| \\
&\leq\; |\gamma_1 - \gamma_0| \, \|DT(\nabla u_0)\|_{L^\infty} \|\nabla v_0\|_{L^\infty} \, |\omega|^{\frac{1}{2}} \, \varepsilon^{\frac{N}{2}} \;\; \|\nabla \tilde{v}_\varepsilon\|_{L^2(\Omega)} \, .
\end{aligned}$$

Let $C := |\omega| \left(\frac{|\gamma_1 - \gamma_0| \|DT(\nabla u_0)\|_{L^\infty} \|\nabla v_0\|_{L^\infty}}{\underline{\gamma} c} \right)^2$. It follows

$$\|\nabla \tilde{v}_\varepsilon\|_{L^2(\Omega)}^2 \;\leq\; C \, \varepsilon^N$$

which is the claimed estimate (4.5.8).

2. The upper bound (4.5.9) to $\|\nabla k_\varepsilon\|_{L^2(\Omega)}^2$ is obtained in the same way.

3. After a change of scale and since by definition $\nabla K \in L^2(\mathbb{R}^N)$ it holds

$$\|\nabla K_\varepsilon\|_{L^2(\Omega)}^2 = \varepsilon^N \int_{\Omega/\varepsilon} |\nabla K|^2 \leq \varepsilon^N \int_{\mathbb{R}^N} |\nabla K|^2 = O(\varepsilon^N)$$

which is estimate (4.5.10).

Proof of Proposition 4.5.4

1. We first study the asymptotic behavior of K and ∇K. The variational form (4.5.6)

$$\int_{\mathbb{R}^N} \gamma DT(U_0)\nabla K.\nabla\eta = -(\gamma_1-\gamma_0)\int_\omega DT(U_0)V_0.\nabla\eta, \quad \forall\,\eta\in\mathcal{H}(\mathbb{R}^N)$$

can be rewritten

$$\int_{\mathbb{R}^N} DT(U_0)\nabla K.\nabla\eta = \left(1-\frac{\gamma_1}{\gamma_0}\right)\int_\omega DT(U_0)(V_0+\nabla K).\nabla\eta, \quad \forall\,\eta\in\mathcal{H}(\mathbb{R}^N). \tag{4.7.58}$$

As matrix $DT(U_0)$ is definite positive, at the price of a linear change of coordinates in $\mathbb{R}^N$, equation (4.7.58) becomes a Laplace equation in $\mathbb{R}^N$ with a source supported by $\partial\omega$. Without loss of generality and for simplicity, we assume that $DT(U_0)=I_N$.

The proof of the asymptotic behavior of K is standard (e.g. [7], §3.1.2). We denote E an elementary solution of the Laplace operator in $\mathbb{R}^N$ given for all $y\in\mathbb{R}^N$, $y\neq 0$ by

$$E(y) := \begin{cases} \frac{1}{2\pi}\ \log|y|, & \text{if } N=2,\\ \frac{1}{(2-N)A^{N-1}}\ |y|^{2-N}, & \text{if } N\geq 3.\end{cases} \tag{4.7.59}$$

In particular

$$\forall N\geq 2, \exists C_N>0, \forall y\in\mathbb{R}^N, y\neq 0, \quad |\nabla E(y)|\leq C_N\,|y|^{1-N}. \tag{4.7.60}$$

Denote $\mathcal{T}$ the distribution in $\mathbb{R}^N$ defined by

$$\langle\mathcal{T},\eta\rangle := \left(\frac{\gamma_1}{\gamma_0}-1\right)\int_\omega (V_0+\nabla K).\nabla\eta, \quad \forall\eta\in C_0^\infty(\mathbb{R}^N).$$

It follows from (4.7.58) that

$$\Delta K = \mathcal{T}.$$

Hence let an element $\tilde{K}$ of the class K given by

$$\tilde{K} = \mathcal{T} * E \tag{4.7.61}$$

Let $\rho>0$ such that $\omega\subset B(0,\rho)$. To study the behavior of $\tilde{K}$ at infinity, let $y\in\mathbb{R}^N$, $|y|\geq 2\rho$. In particular $|z|\,/\,|y|\leq 1/2$, $\forall z\in\omega$.

The convolution (4.7.61) reads

$$\tilde{K}(y) = \left(\frac{\gamma_1}{\gamma_0}-1\right)\int_\omega (V_0+\nabla K(z)).\nabla E(y-z)\;dz. \tag{4.7.62}$$

Since $V_0+\nabla K\in L^2(\omega)$, the Cauchy-Schwarz's inequality yields

$$\left|\tilde{K}(y)\right| \leq C\left(\int_\omega |\nabla E(y-z)|^2\;dz\right)^{\frac{1}{2}},$$

with $C := \left|\frac{\gamma_1}{\gamma_0} - 1\right| \ \|V_0 + \nabla K\|_{L^2(\omega)}$.
In addition, after estimate (4.7.60), it holds

$$\begin{aligned}\int_\omega |\nabla E(y-z)|^2 \ dz &\leq C_N^2 \int_\omega |y-z|^{2-2N} \\ &\leq C_N^2 \int_\omega (|y|-|z|)^{2-2N} \\ &\leq C_N^2 \, |\omega| \left(\frac{1}{2}\right)^{2-2N} |y|^{2-2N} .\end{aligned}$$

Hence

$$\left|\tilde{K}(y)\right| \leq C' \, |y|^{1-N} ,$$

with $C' = C \ C_N \ 2^{N-1} \, |\omega|^{\frac{1}{2}}$. This completes the proof of the claimed asymptotic behavior (4.5.11).
The calculations proving the asymptotic behavior (4.5.12) of ∇K are similar, as differentiating (4.7.61) in the sense of distributions yields

$$\nabla K = \mathcal{T} * \nabla E, \tag{4.7.63}$$

and differentiating twice definition (4.7.59) provides the following control over the matrix norm of the hessian

$$\forall N \geq 2, \exists C'_N > 0, \forall y \in \mathbb{R}^N, y \neq 0, \quad \left\|D^2 E(y)\right\| \leq C'_N \, |y|^{-N} .$$

2. As by definition in holds $\nabla K \in L^2(\mathbb{R}^N)$, the claimed regularity $K \in \mathcal{V}(\mathbb{R}^N)$ is equivalent to $w_p \tilde{K} \in L^p(\mathbb{R}^N)$ and $\nabla K \in L^p(\mathbb{R}^N)$.
   – According to the asymptotic behaviors (4.5.11) of $\tilde{K}$ and (4.5.12) of ∇K, there exist $C > 0$ and $M > 0$ such that $\omega \subset\subset B(0, M)$ and such that

   $$\left|\tilde{K}(y)\right| \leq C \, |y|^{1-N} \ \text{ and } \ |\nabla K(y)| \leq C \, |y|^{-N} , \quad \forall y \in B'_M,$$

   where $B'_M = \mathbb{R}^N \setminus \bar{B}(0, M)$.
   Thus $\tilde{K} \in L^\infty(B'_M)$. As $w_p \in L^p(\mathbb{R}^N)$, it follows that $w_p \tilde{K} \in L^p(B'_M)$. In addition, an integration in spherical coordinates yields

   $$\int_{B'_M} |\nabla K(y)|^p \leq A^{N-1} \ C^p \int_M^\infty r^{-pN+N-1} \, dr \ < +\infty.$$

   Thus $\nabla K \in L^p(B'_M)$.
   – The variational form (4.5.6) defining K can be rewritten in the strong form

   $$\begin{cases} -\operatorname{div}(DT(U_0)K) = 0 & \text{in } \mathbb{R}^N \setminus \partial\omega, \\ \gamma_0 \left(\frac{\partial DT(U_0)K}{\partial n_{out}}\right)_+ - \gamma_1 \left(\frac{\partial DT(U_0)K}{\partial n_{out}}\right)_- = (\gamma_1 - \gamma_0)(DT(U_0)V_0).n_{out} & \text{on } \partial\omega. \end{cases} \tag{4.7.64}$$

   where n_{out} denotes the unit outward normal to $\partial\omega$ and

   $$\left(\frac{\partial DT(U_0)K}{\partial n_{out}}\right)_\pm = \lim_{t \to 0^+} (DT(U_0)\nabla K(x \pm t n_{out})).n_{out}, \quad \forall x \in \partial\omega.$$

Such transmission problems, with a source of zero mean value on $\partial\omega$, have been studied e.g. [5], §2.4. As $\nabla K \in L^2(\mathbb{R}^N)$, K is weakly continuous across $\partial\omega$. The solution is a single layer potential. As $\partial\omega$ is C^2 and the source is continuous, the regularity of the density entails that $K \in L^\infty(B(0,M))$ and $\nabla K \in L^\infty(B(0,M))$. Hence $w_p\tilde{K} \in L^p(B(0,M))$ and $\nabla K \in L^p(B(0,M))$.

One concludes that $w_p\tilde{K} \in L^p(\mathbb{R}^N)$ and $\nabla K \in L^p(\mathbb{R}^N)$. Eventually it holds $K \in \mathcal{V}(\mathbb{R}^N)$.

Proof of Lemma 4.5.5

We aim at proving that

$$\|\nabla k_\varepsilon - \nabla K_\varepsilon\|^2_{L^2(\Omega)} = o(e^N).$$

We start proving a technical lemma. Let a smooth function $\theta : \mathbb{R}^N \to \mathbb{R}$ such that

$$\theta(x) = 0,\ \forall x \in B(0,\rho) \quad \text{and} \quad \theta(x) = 1,\ \forall x \in \mathbb{R}^N \setminus B(0,R)$$

where $0 < \rho < R$ were defined in (4.2.1), that is $\omega \subset\subset B(0,\rho) \subset B(0,R) \subset\subset \Omega\setminus spt(f)$. Denote

$$C_\theta := \sup\left\{\max(|\theta(x)|\,, |\nabla\theta(x)|\,; x \in \mathbb{R}^N\right\} < \infty.$$

Recall function K_ε is defined by

$$K_\varepsilon(x) := \varepsilon\tilde{K}(\varepsilon^{-1}x), \quad \forall x \in \Omega. \tag{4.7.65}$$

Then let the function

$$\kappa_{a\varepsilon} : x \in \Omega \mapsto \theta(x)K_\varepsilon(x).$$

According to the Leibniz formula, for a.e. $x \in \Omega$ it holds

$$|\nabla\kappa_{a\varepsilon}(x)|^2 \le 2C_\theta^2\left(|K_\varepsilon(x)|^2 + |\nabla K_\varepsilon(x)|^2\right).$$

Since $K_\varepsilon \in H^1(\Omega)$, it follows $\kappa_{a\varepsilon} \in H^1(\Omega)$.

Lemma 4.7.3. *It holds*

$$\|\nabla\kappa_{a\varepsilon}\|^2_{L^2(\Omega)} \quad = \quad o(\varepsilon^N). \tag{4.7.66}$$

Proof. 1. In $B(0,\rho)$, it holds $\theta = 0$. Thus

$$\int_{B(0,\rho)} |\nabla\kappa_{a\varepsilon}|^2 = 0. \tag{4.7.67}$$

2. Integrating in $B(0,R)\setminus B(0,\rho)$, according to the asymptotic behavior of K given by (4.5.11) and since $\nabla K \in L^2(\mathbb{R}^N)$ one obtains

$$\begin{aligned}
\frac{1}{2C_\theta^2}\int_{B(0,R)\setminus B(0,\rho)} |\nabla\kappa_{a\varepsilon}|^2 &\le \int_{B(0,R)\setminus B(0,\rho)} |K_\varepsilon|^2 \ + \int_{B(0,R)\setminus B(0,\rho)} |\nabla K_\varepsilon|^2 \\
&\le \varepsilon^{2+N}\int_{B(0,R/\varepsilon)\setminus B(0,\rho/\varepsilon)} \left|\tilde{K}\right|^2 \ + \ \varepsilon^N\int_{B(0,R/\varepsilon)\setminus B(0,\rho/\varepsilon)} |\nabla K|^2 \\
&\le \varepsilon^{N+2}\ 0\left(\left(\frac{\rho}{\varepsilon}\right)^{2-2N}\left(\frac{R}{\varepsilon}\right)^N\right) \ + \varepsilon^N\int_{\mathbb{R}^N\setminus B(0,\rho/\varepsilon)} |\nabla K|^2 \\
&\le \varepsilon^N\left(O\left(\varepsilon^N\right) + o(1)\right) = o(\varepsilon^N).
\end{aligned} \tag{4.7.68}$$

3. Lastly it holds $\kappa_{a\varepsilon} = K_\varepsilon$ in $\Omega \setminus B(0,R)$. Again $\nabla K \in L^2(\mathbb{R}^N)$ and thus

$$\int_{\Omega\setminus B(0,R)} |\nabla \kappa_{a\varepsilon}|^2 \leq \varepsilon^N \int_{\mathbb{R}^N\setminus B(0,R/\varepsilon)} |\nabla K|^2 = o\left(\varepsilon^N\right). \tag{4.7.69}$$

Gathering (4.7.67), (4.7.68) and (4.7.69), one obtains the claimed estimate (4.7.66).
□

We now prove Lemma 4.5.5.

1. We begin proving estimate (4.5.15). For all $\eta \in \mathcal{H}$, we define $\eta_1 \in \mathcal{H}(\mathbb{R}^N)$ by

$$\eta_1(y) := \frac{1}{\varepsilon}\eta(\varepsilon y),\ \forall y \in \Omega/\varepsilon \quad \text{and} \quad \eta_1(y) := 0,\ \forall y \in \mathbb{R}^N \setminus (\Omega/\varepsilon).$$

Applying the variational form (4.5.6) to $\eta_1 \in \mathcal{H}(\mathbb{R}^N)$ and making the change of scale backward, one obtains

$$\int_\Omega \gamma_\varepsilon DT(U_0)\nabla K_\varepsilon.\nabla\eta = -\left(\gamma_1-\gamma_0\right)\int_{\omega_\varepsilon} DT(U_0)V_0.\nabla\eta,\ \forall \eta \in \mathcal{H}.$$

Then calculating the difference with the variational form (4.5.5) yields

$$\int_\Omega \gamma_\varepsilon DT(U_0)(\nabla k_\varepsilon - \nabla K_\varepsilon).\nabla\eta = 0,\ \forall \eta \in \mathcal{H}. \tag{4.7.70}$$

Recall function $\kappa_{a\varepsilon}$. It holds $\kappa_{a\varepsilon} \in H^1(\Omega)$ and $K_\varepsilon - \kappa_{a\varepsilon} \in \mathcal{H}$. Choosing $\eta = k_\varepsilon - (K_\varepsilon - \kappa_{a\varepsilon}) \in \mathcal{H}$ and plugging into (4.7.70) it holds

$$\int_\Omega \gamma_\varepsilon DT(U_0)(\nabla k_\varepsilon - \nabla K_\varepsilon).(\nabla k_\varepsilon - \nabla K_\varepsilon) = -\int_\Omega \gamma_\varepsilon DT(U_0)(\nabla k_\varepsilon - \nabla K_\varepsilon).\nabla\kappa_{a\varepsilon}.$$

Then applying condition (4) one obtains

$$\begin{aligned}
\underline{\gamma}\ c\int_\Omega |\nabla k_\varepsilon - \nabla K_\varepsilon|^2 &\leq \int_\Omega \gamma_\varepsilon DT(U_0)(\nabla k_\varepsilon - \nabla K_\varepsilon).(\nabla k_\varepsilon - \nabla K_\varepsilon) \\
&\leq \left|\int_\Omega \gamma_\varepsilon DT(U_0)(\nabla k_\varepsilon - \nabla K_\varepsilon).\nabla\kappa_{a\varepsilon}\right| \\
&\leq \overline{\gamma}\left\|DT(U_0)\right\|\left(\int_\Omega |\nabla k_\varepsilon - \nabla K_\varepsilon|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla\kappa_{a\varepsilon}|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

That is

$$\left\|\nabla k_\varepsilon - \nabla K_\varepsilon\right\|^2_{L^2(\Omega)} \leq \left(\overline{\gamma}\left\|DT(U_0)\right\|/\underline{\gamma}\ c\right)^2 \left\|\nabla\kappa_{a\varepsilon}\right\|^2_{L^2(\Omega)}.$$

Since after estimate (4.7.66) it holds

$$\left\|\nabla\kappa_{a\varepsilon}\right\|^2_{L^2(\Omega)} = o\left(\varepsilon^N\right),$$

it follows that

$$\left\|\nabla k_\varepsilon - \nabla K_\varepsilon\right\|^2_{L^2(\Omega)} = o\left(\varepsilon^N\right),$$

which completes the proof of (4.5.15).

2. Let us now prove estimate (4.5.16). Let $\alpha > 0$ and $r \in (0,1)$.
   By convexity
$$|\nabla k_\varepsilon|^2 \leq 2\,|\nabla k_\varepsilon - \nabla K_\varepsilon|^2 + 2\,|\nabla K_\varepsilon|^2 .$$
   After a change of scale one obtains
$$\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon|^2 \leq 2\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon - \nabla K_\varepsilon|^2 + 2\,\varepsilon^N \int_{\mathbb{R}^N\setminus B(0,\alpha\varepsilon^{r-1})} |\nabla K|^2 . \quad (4.7.71)$$
   After (4.5.15) it holds
$$\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon - \nabla K_\varepsilon|^2 \leq \int_\Omega |\nabla k_\varepsilon - \nabla K_\varepsilon|^2 = o(\varepsilon^N).$$
   Again $\nabla K \in L^2(\mathbb{R}^N)$ and $r-1<0$ entail that
$$\int_{\mathbb{R}^N\setminus B(0,\alpha\varepsilon^{r-1})} |\nabla K|^2 = o(1).$$
   Hence (4.7.71) implies
$$\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon|^2 \;=\; o(\varepsilon^N).$$

Proof of Lemma 4.5.6

Let us prove estimate (4.5.17).

After Lemma 4.4.1, ∇u_0 is β-Hölder continuous at point $x_0 = 0$ for some $\beta > 0$. After condition (1), DT is $\tilde{\alpha}$-Hölder continuous at point U_0 for some $\tilde{\alpha} > 0$. After Lemma 4.4.1 and condition (1), it holds $DT(\nabla u_0) \in L^\infty(\Omega)$. After Lemma 4.5.1 or according to the regularity assumption made about ∇v_0, ∇v_0 is $\tilde{\beta}$-Hölder continuous at point $x_0 = 0$ for some $\tilde{\beta} > 0$.

Let $\tilde{\tau} := \min(\tilde{\alpha}\beta, \tilde{\beta}) > 0$. Hence there exist $\delta \in (0,1)$ and $L > 0$ such that for all $x \in B(0,\delta)$ it holds
$$\|DT(\nabla u_0(x)) - DT(U_0)\| + |DT(\nabla u_0(x))\nabla v_0(x) - DT(U_0)V_0| \leq L\,|x|^{\tilde{\tau}} . \quad (4.7.72)$$
Let $\rho > 0$ such that $\omega \subset B(0,\rho)$ (see (4.2.1)). So as to apply estimate (4.5.16), we choose $\alpha := \rho$ and $r := 1/2$. Lastly for all $\varepsilon \in (0, \min(1, (\delta/\rho)^2)$ it holds
$$\omega_\varepsilon \subset B(0,\rho\varepsilon) \subset B(0,\rho\varepsilon^r) \subset B(0,\delta).$$

We can now start our estimations. According to condition (4), it holds
$$\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|^2_{L^2(\Omega)} = \int_\Omega |\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon|^2 \leq \frac{1}{\underline{\gamma} c}\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon)^2 . \quad (4.7.73)$$

Calculating the difference between the variational forms (4.5.4) and (4.5.5) and choosing $\eta = \tilde{v}_\varepsilon - k_\varepsilon \in \mathcal{H}$, one obtains:
$$\begin{aligned}
&\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon)^2 \\
&= -(\gamma_1-\gamma_0)\int_{\omega_\varepsilon} (DT(\nabla u_0)\nabla v_0 - DT(U_0)V_0)\,.\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon) \\
&\qquad - \int_\Omega (\gamma_\varepsilon DT(\nabla u_0) - \gamma_\varepsilon DT(U_0))\,\nabla k_\varepsilon.\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon)\,. \quad (4.7.74)
\end{aligned}$$

1. Regarding the first term on the right-hand side of (4.7.74), it follows from continuity inequality (4.7.72) and from Cauchy-Schwarz's inequality that

$$\begin{aligned}&\left|\int_{\omega_\varepsilon} (DT(\nabla u_0)\nabla v_0 - DT(U_0)V_0)\,.\,(\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon)\right| \\ &\quad \leq L\rho^{\tilde{\tau}}\varepsilon^{\tilde{\tau}}\,|\omega|^{\frac{1}{2}}\,\varepsilon^{\frac{N}{2}}\,\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} = C_1\varepsilon^{\tilde{\tau}+\frac{N}{2}}\,\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} \end{aligned} \tag{4.7.75}$$

   where C_1 is a constant.

2. Then consider the second term on the right side of (4.7.74). We split the integral into $B(0,\alpha\varepsilon^r)$ and into $\Omega \setminus B(0,\alpha\varepsilon^r)$.

   Applying the continuity inequality (4.7.72), the Cauchy-Schwarz's inequality and the upper-bound (4.5.9) about $\|\nabla k_\varepsilon\|_{L^2(\Omega)}$, it follows

$$\begin{aligned}&\left|\int_{B(0,\alpha\varepsilon^r)} (\gamma_\varepsilon DT(\nabla u_0) - \gamma_\varepsilon DT(U_0))\,\nabla k_\varepsilon.\,(\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon)\right| \\ &\quad \leq \overline{\gamma}L\alpha^{\tilde{\tau}}\varepsilon^{r\tilde{\tau}}C^{\frac{1}{2}}\varepsilon^{\frac{N}{2}}\,\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} = C_2\varepsilon^{r\tilde{\tau}+\frac{N}{2}}\,\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)}\end{aligned} \tag{4.7.76}$$

   where C_2 is a constant.

   Regarding the integral in $\Omega \setminus B(0,\alpha\varepsilon^r)$, the term $\gamma_\varepsilon DT(\nabla u_0) - \gamma_\varepsilon DT(U_0)$ is bounded by $\tilde{C} := 2\overline{\gamma}\,\|DT(\nabla u_0)\|_{L^\infty(\Omega)}$. After Cauchy-Schwarz's inequality and estimate (4.5.16) one obtains

$$\begin{aligned}&\left|\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} (\gamma_\varepsilon DT(\nabla u_0) - \gamma_\varepsilon DT(U_0))\,\nabla k_\varepsilon.\,(\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon)\right| \\ &\leq \tilde{C}\left(\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon|^2\right)^{\frac{1}{2}}.\,\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} \\ &\qquad = o(\varepsilon^{\frac{N}{2}})\;\;\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)}.\end{aligned} \tag{4.7.77}$$

Therefore, gathering (4.7.73), (4.7.74), (4.7.75), (4.7.76) and (4.7.77) and dividing by $\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)}$, it follows that

$$\begin{aligned}\underline{\gamma}c\,\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} &\leq C_1\varepsilon^{\tilde{\tau}+\frac{N}{2}} + C_2\varepsilon^{r\tilde{\tau}+\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}) \\ &= o(\varepsilon^{\frac{N}{2}}).\end{aligned}$$

Powering to the square, one obtains

$$\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|^2_{L^2(\Omega)} \quad = \quad o(e^N)$$

which is the claimed estimate (4.5.17).

4.7.4 Proofs about the topological asymptotic expansion

Proof of Lemma 4.6.1

It follows from definitions (4.6.8) and (4.6.11) that

$$\begin{aligned}\tilde{j}_1(\varepsilon) - \varepsilon^N J_1 &= (\gamma_1 - \gamma_0)\left[\int_{\omega_\varepsilon} T(U_0).(V_0 + \nabla k_\varepsilon) - \varepsilon^N \int_\omega T(U_0).(V_0 + \nabla K)\right] \\ &= (\gamma_1 - \gamma_0)\; T(U_0).\int_{\omega_\varepsilon} (\nabla k_\varepsilon - \nabla K_\varepsilon).\end{aligned}$$

Hence after estimate (4.5.15) and Cauchy-Schwarz's inequality, it holds

$$\begin{aligned}\left|\tilde{j}_1(\varepsilon) - \varepsilon^N J_1\right| &\leq 2\overline{\gamma}\, |T(U_0)|\, |\omega|^{\frac{1}{2}}\, \varepsilon^{\frac{N}{2}}\, \|\nabla k_\varepsilon - \nabla K_\varepsilon\|_{L^2(\Omega)} \\ &\leq O(\varepsilon^{\frac{N}{2}})\; o(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^N)\end{aligned}$$

which completes the proof of Lemma 4.6.1.

Proof of Lemma 4.6.2

It follows from definitions (4.6.6) and (4.6.8) that

$$\begin{aligned}j_1(\varepsilon) - \tilde{j}_1(\varepsilon) &= (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} \left[T(\nabla u_0).\nabla v_\varepsilon - T(U_0).(V_0 + \nabla k_\varepsilon)\right] \\ &= (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(\nabla u_0).\nabla v_0 - T(U_0).V_0 \\ &\quad +(\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(\nabla u_0).\nabla \tilde{v}_\varepsilon - T(U_0).\nabla k_\varepsilon.\end{aligned}$$

Since $x \in \Omega \mapsto T(\nabla u_0(x)).\nabla v_0(x)$ is continuous at point $x_0 = 0$, it holds

$$\int_{\omega_\varepsilon} T(\nabla u_0).\nabla v_0 - T(U_0).V_0 = |\omega_\varepsilon|\, o(1) = o(\varepsilon^N).$$

Moreover, since $x \in \Omega \mapsto T(\nabla u_0(x))$ is continuous at point $x_0 = 0$, after Cauchy-Schwarz's inequality and estimates (4.5.17) and (4.5.9), it holds

$$\begin{aligned}&\int_{\omega_\varepsilon} |T(\nabla u_0).\nabla \tilde{v}_\varepsilon - T(U_0).\nabla k_\varepsilon| \\ &\leq \int_{\omega_\varepsilon} |T(\nabla u_0)|\, |\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon| + \int_{\omega_\varepsilon} |T(\nabla u_0) - T(U_0)|\, |\nabla k_\varepsilon| \\ &\leq |\omega|^{\frac{1}{2}}\, \varepsilon^{\frac{N}{2}} \left(\|T(\nabla u_0)\|_\infty\, \|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} + o(1)\, \|\nabla k_\varepsilon\|_{L^2(\Omega)}\right) \\ &\qquad = O(\varepsilon^{\frac{N}{2}})\left(o(\varepsilon^{\frac{N}{2}}) + o(1)\; O(\varepsilon^{\frac{N}{2}})\right) = o(\varepsilon^N).\end{aligned}$$

Thus

$$j_1(\varepsilon) - \tilde{j}_1(\varepsilon) = o(\varepsilon^N)$$

which completes the proof of Lemma 4.6.2.

Proof of Lemma 4.6.4

Recall after definition (4.6.15) it holds

$$\tilde{j}_2(\varepsilon) \;=\; \int_\Omega \gamma_\varepsilon S_{U_0}(\nabla h_\varepsilon).V_0 + (\gamma_1-\gamma_0)\int_{\omega_\varepsilon}\left[DT(U_0)V_0.\nabla h_\varepsilon - T(U_0).\nabla k_\varepsilon\right].$$

Regarding J_2 after definition (4.6.16) it holds

$$J_2 \;=\; \int_{\mathbb{R}^N} \gamma S_{U_0}(\nabla H).V_0 + (\gamma_1-\gamma_0)\int_{\omega}\left[DT(U_0)V_0.\nabla H - T(U_0).\nabla K\right].$$

Then making the change of scale backward,

$$\begin{aligned}\varepsilon^N J_2 \;=\; &\varepsilon^N \int_{\mathbb{R}^N\setminus(\Omega/\varepsilon)} \gamma S_{U_0}(\nabla H).V_0 + \int_\Omega \gamma_\varepsilon S_{U_0}(\nabla H_\varepsilon).V_0\\ &+(\gamma_1-\gamma_0)\int_{\omega_\varepsilon}\left[DT(U_0)V_0.\nabla H_\varepsilon - T(U_0).\nabla K_\varepsilon\right].\end{aligned}$$

The first integral on the right-hand side is the remainder of a converging integral. Thus

$$\int_{\mathbb{R}^N\setminus(\Omega/\varepsilon)} \gamma S_{U_0}(\nabla H).V_0 \;=\; o(1).$$

It follows

$$\begin{aligned}&\varepsilon^N J_2 - o(\varepsilon^N) =\\ &\qquad\int_\Omega \gamma_\varepsilon S_{U_0}(\nabla H_\varepsilon).V_0 + (\gamma_1-\gamma_0)\int_{\omega_\varepsilon}\left[DT(U_0)V_0.\nabla H_\varepsilon - T(U_0).\nabla K_\varepsilon\right].\end{aligned} \tag{4.7.78}$$

Therefore gathering (4.6.15) and (4.7.78) yields

$$\begin{aligned}\tilde{j}_2(\varepsilon) - \varepsilon^N J_2 - o(\varepsilon^N) \;:=\; &\int_\Omega \gamma_\varepsilon\left[S_{U_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla H_\varepsilon)\right].V_0\\ &+(\gamma_1-\gamma_0)\int_{\omega_\varepsilon} DT(U_0)V_0.(\nabla h_\varepsilon - \nabla H_\varepsilon) \qquad (4.7.79)\\ &-(\gamma_1-\gamma_0)\int_{\omega_\varepsilon} T(U_0).(\nabla k_\varepsilon - \nabla K_\varepsilon). \qquad (4.7.80)\end{aligned}$$

Regarding the second term (4.7.79) on the right-hand side, Hölder's inequality and estimate (4.4.21) imply

$$\begin{aligned}\int_{\omega_\varepsilon} DT(U_0)V_0.(\nabla h_\varepsilon - \nabla H_\varepsilon) \;&\le\; |DT(U_0)V_0|\,|\omega|^{\frac{1}{q}}\,\varepsilon^{\frac{N}{q}}\,\|\nabla h_\varepsilon - \nabla H_\varepsilon\|_{L^p(\Omega)}\\ &=\; O(\varepsilon^{\frac{N}{q}})\; o(\varepsilon^{\frac{N}{p}}) = o(\varepsilon^N).\end{aligned}$$

Similarly for the third term (4.7.80) on the right-hand side, Cauchy-Schwarz's inequality and estimate (4.5.15) entail

$$\begin{aligned}\int_{\omega_\varepsilon} T(U_0).(\nabla k_\varepsilon - \nabla K_\varepsilon) \;&\le\; |T(U_0)|\,|\omega|^{\frac{1}{2}}\,\varepsilon^{\frac{N}{2}}\,\|\nabla k_\varepsilon - \nabla K_\varepsilon\|_{L^2(\Omega)}\\ &=\; O(\varepsilon^{\frac{N}{2}})\; o(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^N).\end{aligned}$$

It follows

$$\tilde{j}_2(\varepsilon) - \varepsilon^N J_2 \quad := \quad \int_\Omega \gamma_\varepsilon \left[S_{U_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla H_\varepsilon) \right] .V_0 \; + \; o(\varepsilon^N). \tag{4.7.81}$$

Condition (7) reads

$$\begin{aligned}
&\int_\Omega |S_{U_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla H_\varepsilon)| \\
&\leq \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon| \right) \left[c_0 + c_{p-3} \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon| \right)^{p-3} \right] \\
&= c_0 \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon| \right) \\
&\qquad\qquad\qquad + c_{p-3} \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon| \right)^{p-2},
\end{aligned}$$

with $c_{p-3} = 0$ for all $p \in [2, 3]$.

Hence it follows from estimates (4.4.26) and (4.4.27) of Proposition 4.4.12 that

$$\int_\Omega |S_{U_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla H_\varepsilon)| \quad = \quad o(\varepsilon^N).$$

Therefore estimate (4.7.81) entails

$$\tilde{j}_2(\varepsilon) - \varepsilon^N J_2 \quad = o(\varepsilon^N) \tag{4.7.82}$$

which completes the proof of Lemma 4.6.4.

Proof of Lemma 4.6.5

1. We first prove estimate (4.6.19). Since ∇v_0 is $\tilde{\beta}$-Hölder continuous at point $x_0 = 0$ for some $\tilde{\beta} > 0$, there exist $\delta > 0$ and $L > 0$ such that

$$|\nabla v_0(x) - V_0| \leq L \, |x|^{\tilde{\beta}}, \quad \forall x \in B(0, \delta).$$

To apply estimate (4.4.22), we choose $\alpha := \delta$ and $r := 1/2$. Hence for all $\varepsilon \in (0, 1)$, according to estimates (4.4.10) and (4.4.22) it follows

$$\begin{aligned}
&\int_\Omega |\nabla v_0 - V_0| \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2 \right) \\
&\leq \int_{B(0,\alpha\varepsilon^r)} L \, |x|^{\tilde{\beta}} \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2 \right) + 2 \, \|\nabla v_0\|_{L^\infty(\Omega)} \int_{\Omega \setminus B(0,\alpha\varepsilon^r)} \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2 \right) \\
&\qquad\qquad\qquad \leq L\alpha^{\tilde{\beta}} \varepsilon^{r\tilde{\beta}} \; O(\varepsilon^N) + o(\varepsilon^N) = o(\varepsilon^N),
\end{aligned}$$

which completes the proof of estimate (4.6.19).

2. For all $p \in (3, \infty)$ and for all $\lambda \in \mathbb{R}_+$ it holds $\lambda^{p-1} \leq \lambda^2 + \lambda^p$. Hence the claimed (4.6.20) estimate

$$\forall \; p \in (3, \infty), \int_\Omega |\nabla v_0 - V_0| \, |\nabla h_\varepsilon|^{p-1} = o(\varepsilon^N)$$

follows immediately from estimate (4.6.19).

Proof of Lemma 4.6.6

Recall definition (4.6.7)

$$j_2(\varepsilon) \;=\; \int_\Omega \gamma_\varepsilon S_{\nabla u_0}(\nabla \tilde{u}_\varepsilon).\nabla v_0 + (\gamma_1-\gamma_0)\int_{\omega_\varepsilon}\left[DT(\nabla u_0)\nabla v_0.\nabla\tilde{u}_\varepsilon - T(\nabla u_0).\nabla\tilde{v}_\varepsilon\right].$$

Calculating the difference between (4.6.7) and (4.6.15) yields

$$\begin{aligned} j_2(\varepsilon)-\tilde{j}_2(\varepsilon) \;=\; & \int_\Omega \gamma_\varepsilon\left[S_{\nabla u_0}(\nabla\tilde{u}_\varepsilon).\nabla v_0 - S_{U_0}(\nabla h_\varepsilon).V_0\right] \\ & +(\gamma_1-\gamma_0)\int_{\omega_\varepsilon}\left[DT(\nabla u_0)\nabla v_0.\nabla\tilde{u}_\varepsilon - DT(U_0)V_0.\nabla h_\varepsilon\right] \\ & -(\gamma_1-\gamma_0)\int_{\omega_\varepsilon}\left[T(\nabla u_0).\nabla\tilde{v}_\varepsilon - T(U_0).\nabla k_\varepsilon\right]. \end{aligned} \tag{4.7.83}$$

Let $\delta>0$. Due to the continuity of ∇u_0 and ∇v_0 at point $x_0=0$ and to the continuity of DT, for $\varepsilon>0$ small enough it holds

$$\max(|DT(\nabla u_0)\nabla v_0 - DT(U_0)V_0|\,, |T(\nabla u_0)-T(U_0)|)\le\delta \ \text{ in } \omega_\varepsilon.$$

Hence after Cauchy-Schwarz's inequality and estimates (4.4.28) and (4.4.10)

$$\begin{aligned} & \int_{\omega_\varepsilon}|DT(\nabla u_0)\nabla v_0.\nabla\tilde{u}_\varepsilon - DT(U_0)V_0.\nabla h_\varepsilon| \\ & \le \int_{\omega_\varepsilon}|DT(\nabla u_0)\nabla v_0|\,|\nabla\tilde{u}_\varepsilon-\nabla h_\varepsilon| + \int_{\omega_\varepsilon}|DT(\nabla u_0)\nabla v_0 - DT(U_0)V_0|\,|\nabla h_\varepsilon| \\ & \le |\omega|^{\frac12}\,\varepsilon^{\frac N2}\left[\|DT(\nabla u_0)\nabla v_0\|_\infty\,\|\nabla\tilde{u}_\varepsilon-\nabla h_\varepsilon\|_{L^2(\Omega)} + \delta\,\|\nabla h_\varepsilon\|_{L^2(\Omega)}\right] \\ & \qquad\qquad \le O(\varepsilon^{\frac N2})\ o(\varepsilon^{\frac N2}) + \ O(\varepsilon^{\frac N2})\ \delta\ O(\varepsilon^{\frac N2}) = o(\varepsilon^N). \end{aligned}$$

Similarly after Cauchy-Schwarz's inequality and estimates (4.5.17) and (4.5.9)

$$\begin{aligned} & \int_{\omega_\varepsilon}|T(\nabla u_0).\nabla\tilde{v}_\varepsilon - T(U_0).\nabla k_\varepsilon| \\ & \le \int_{\omega_\varepsilon}|T(\nabla u_0)|\,|\nabla\tilde{v}_\varepsilon-\nabla k_\varepsilon| + \int_{\omega_\varepsilon}|T(\nabla u_0)-T(U_0)|\,|\nabla k_\varepsilon| \\ & \le |\omega|^{\frac12}\,\varepsilon^{\frac N2}\left[\|T(\nabla u_0)\|_{L^\infty(\Omega)}\,\|\nabla\tilde{v}_\varepsilon-\nabla k_\varepsilon\|_{L^2(\Omega)} + \delta\,\|\nabla k_\varepsilon\|_{L^2(\Omega)}\right] \\ & \qquad\qquad \le O(\varepsilon^{\frac N2})\ o(\varepsilon^{\frac N2}) + \ O(\varepsilon^{\frac N2})\ \delta\ O(\varepsilon^{\frac N2}) = o(\varepsilon^N). \end{aligned}$$

Thus (4.7.83) yields

$$j_2(\varepsilon)-\tilde{j}_2(\varepsilon)-o(\varepsilon^N) \;=\; \int_\Omega\gamma_\varepsilon\left[S_{\nabla u_0}(\nabla\tilde{u}_\varepsilon).\nabla v_0 - S_{U_0}(\nabla h_\varepsilon).V_0\right], \tag{4.7.84}$$

which one can split into three terms

$$\begin{aligned} & \int_\Omega\gamma_\varepsilon\left[S_{\nabla u_0}(\nabla\tilde{u}_\varepsilon).\nabla v_0 - S_{U_0}(\nabla h_\varepsilon).V_0\right] \\ & = \int_\Omega\gamma_\varepsilon\left[S_{\nabla u_0}(\nabla\tilde{u}_\varepsilon) - S_{\nabla u_0}(\nabla h_\varepsilon)\right].\nabla v_0 + \int_\Omega\gamma_\varepsilon\left[S_{\nabla u_0}(\nabla h_\varepsilon)-S_{U_0}(\nabla h_\varepsilon)\right].\nabla v_0 \\ & \qquad\qquad + \int_\Omega\gamma_\varepsilon S_{U_0}(\nabla h_\varepsilon).\left(\nabla v_0 - V_0\right). \end{aligned} \tag{4.7.85}$$

1. Regarding the first term on the right-hand side of (4.7.85), as $\nabla u_0 \in L^\infty(\Omega)$, it follows from condition (7) that

$$\int_\Omega |S_{\nabla u_0}(\nabla\tilde{u}_\varepsilon) - S_{\nabla u_0}(\nabla h_\varepsilon)| \leq \int_\Omega |\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla\tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right) \left[c_0 + c_{p-3}\left(|\nabla\tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right)^{p-3}\right]$$

with $c_{p-3} = 0$ for all $p \in [2,3]$. Thus estimates (4.4.29) and (4.4.30) of Proposition 4.4.13 entail that

$$\int_\Omega |S_{\nabla u_0}(\nabla\tilde{u}_\varepsilon) - S_{\nabla u_0}(\nabla h_\varepsilon)| = o(\varepsilon^N).$$

As $\nabla v_0 \in L^\infty(\Omega)$, it follows

$$\int_\Omega \gamma_\varepsilon \left[S_{\nabla u_0}(\nabla\tilde{u}_\varepsilon) - S_{\nabla u_0}(\nabla h_\varepsilon)\right].\nabla v_0 = o(\varepsilon^N).$$

2. Regarding the second term on the right-hand side of (4.7.85), as $\nabla u_0 \in L^\infty(\Omega)$, according to condition (8)

$$\int_\Omega |S_{\nabla u_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla h_\varepsilon)| \leq \int_\Omega |\nabla u_0 - U_0| \left[d_0 |\nabla h_\varepsilon|^2 + d_{p-4} |\nabla h_\varepsilon|^{p-2}\right]$$

with $d_{p-4} = 0$ for all $p \in [2,4]$. Thus estimates (4.4.23) and (4.4.24) of Proposition 4.4.12 entail that

$$\int_\Omega |S_{\nabla u_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla h_\varepsilon)| = o(\varepsilon^N).$$

As $\nabla v_0 \in L^\infty(\Omega)$, it follows

$$\int_\Omega \gamma_\varepsilon \left[S_{\nabla u_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla h_\varepsilon)\right].\nabla v_0 = o(\varepsilon^N).$$

3. Regarding the third term on the right-hand side of (4.7.85), according to (4.1.2) derived from condition (7), it holds

$$\int_\Omega |S_{U_0}(\nabla h_\varepsilon)|\ |\nabla v_0 - V_0| \leq \int_\Omega |\nabla v_0 - V_0| \left[c_0 |\nabla h_\varepsilon|^2 + c_{p-3} |\nabla h_\varepsilon|^{p-1}\right]$$

with $c_{p-3} = 0$ for all $p \in [2,3]$. Hence it follows from estimates (4.6.19) and (4.6.20) that

$$\int_\Omega |S_{U_0}(\nabla h_\varepsilon)|\ |\nabla u_0 - U_0| = o(\varepsilon^N).$$

Hence one concludes from estimates here above of the three terms on the right-hand side of (4.7.85) that

$$\int_\Omega \gamma_\varepsilon \left[S_{\nabla u_0}(\nabla\tilde{u}_\varepsilon).\nabla v_0 - S_{U_0}(\nabla h_\varepsilon).V_0\right] = o(\varepsilon^N).$$

Therefore equation (4.7.84) yields

$$j_2(\varepsilon) - \tilde{j}_2(\varepsilon) = o(\varepsilon^N) \tag{4.7.86}$$

which completes the proof of Lemma 4.6.6.

4.8 Conclusion

In Part I, we first analyzed the specific issues arising in the process of obtaining a topological asymptotic expansion for a second order quasilinear elliptic equation, by comparison with a linear elliptic equation.

When trying to define the variation of the direct state at scale 1 in $\mathbb{R}^N$, it turns out that this variation can be defined by applying the Minty-Browder theorem to a specific nonlinear operator, which is derived from the considered quasilinear equation. The requirements of the Minty-Browder theorem bring into light a two-norm discrepancy involving the L^p and the L^2 norms of the gradient. They require to consider both

- a functional space which is equipped with a norm giving control on both the L^p and the L^2 norms of the gradient and which enjoys in addition a Poincaré inequality;
- and a quasilinear elliptic equation such that resulting nonlinear operator enjoys both p- and 2- ellipticity properties, which is not the case for the p-Laplace equation.

The first condition justifies that we built the quotient weighted Sobolev space $\mathcal{V}(\mathbb{R}^n)$ and the quotient weighted Hilbert space $\mathcal{H}(\mathbb{R}^N)$ in chapter 3.

The second condition explains why in chapter 4 we built a class of quasilinear equations for which the resulting nonlinear operator enjoys both p- and 2- ellipticity properties.

On top of giving a proper definition to the variation of the direct state at scale 1, several other key features of the linear method had to be adapted to the nonlinear case. In particular, implementing the method required to:

1. ensure duality between the variation of the direct state and the corresponding variation of the adjoint state, at each stage of approximation;
2. determine the asymptotic behavior of the variation of the direct state at scale 1;
3. determine with respect to the variation of the direct state, what does mean 'far away from the perturbation' by opposition to 'close to the perturbation'.

As a result, our main contribution is Theorem 4.3.1 which provides the topological asymptotic expansion for quasilinear elliptic equations of the considered class.

The process of obtaining this result has been quite intricate, as the patient reader could check it. As several components were not directly available from the literature, they had to be specifically built or proven: function spaces, class of quasilinear equations, definition and study of the variations of the direct state, asymptotic behavior of the variation at scale 1 in $\mathbb{R}^N$, estimates allowing the final obtainment of the topological asymptotic expansion. More importantly all these components had to match each others.

But our hope is that the doorway of topological asymptotic expansions for quasilinear elliptic equations is now opened. As topological asymptotic expansions will gradually become available for larger classes of nonlinear equations and of functionals, the scope of attainable applicative tasks should significantly broaden, in particular in shape optimization and in imaging.

Further research can now be extended in several directions, for instance:

- a first goal is to obtain similar topological asymptotic expansions for larger classes of quasilinear elliptic equations, in particular for degenerate equations such as the p-Laplace equation. This will require to deepen the study of chapter 2 about issues raised by a nonlinear equation and then to build an appropriate functional setting;
- another goal will be to extend the method to the equations of nonlinear elasticity;
- in terms of numerical applications, an important issue will be to assess the cost of computing the topological gradient $x_0 \in \Omega \to g(x_0)$ stated in Theorem 4.3.1, given the fact that H is solution of a nonlinear transmission equation in $\mathbb{R}^N$.

Part II

Estimates and asymptotic expansions of condenser p-capacities

Notation for Part II

In all Part II, let $p \in (1, +\infty)$.

Let $N \in \mathbb{N}$. In sections 5.1, 5.2 and 5.3, we assume $N \geq 1$. Whenever segments are considered, that is in section 5.4 as in the proof of Proposition 5.4.6 in subsection 5.5.2, we assume $N \geq 2$.

Let a bounded domain $\Omega \subset \mathbb{R}^N$ and a compact subset $K \subset\subset \Omega$.

Classical notation will be used as follows:

1. The symbol $|E|$ denotes either the usual euclidean norm of E in $\mathbb{R}^N$ when $E \in \mathbb{R}^N$, or the N-dimensional Lebesgue measure of E when $E \subset \mathbb{R}^N$.
2. S^{N-1} will be the unit sphere in $\mathbb{R}^N$ and A^{N-1} its surface area.
3. $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support $\subset \Omega$ and $\mathcal{D}'(\Omega)$ denotes the space of distributions in Ω.
4. We denote $W^{1,p}(\Omega)$ the usual Sobolev space defined by

$$W^{1,p}(\Omega) := \{u \in \mathcal{D}'(\Omega); u \in L^p(\Omega), \nabla u \in L^p(\Omega)\}$$

endowed with the norm defined by

$$\|u\|_{1,p} := \left[\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\right]^{\frac{1}{p}}.$$

5. We denote $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.
6. For simplicity, we denote $\beta := (p - N)/(p - 1) \in (-\infty, 1]$. It is convenient to remember that $p > N \Leftrightarrow \beta > 0$ and that $\beta < 1$, for all $N \geq 2$.

Chapter 5

Estimates and asymptotic expansions of condenser p-capacities

5.1 Introduction

5.1.1 Context of chapter 5

The concept of capacity originated from the physics of electrostatic condensers. It has since then been widely extended on the mathematical side as set functions in the linear potential theory [29] and more recently in the nonlinear potential theory (Maz'ya [65, 67], Adams & Hedberg [1], Heinonen, Kilpeläinen & Martio [49], Turesson [88] ...). Many types of capacities were studied. An axiomatic theory of capacities was set by Choquet in the 1950's ([36] or [39], Appendix II).

Let $p \in (1, \infty)$. The *variational p-capacity* of a compact set K, denoted $c_p(K)$, is defined as the infimum deviation of energy that is caused by an inequality constraint required in compact K. More precisely, one defines

$$c_p(K) := \inf \left\{ \int_{\mathbb{R}^N} |u|^p + |\nabla u|^p \ ; \ u \in C_0^\infty(\mathbb{R}^N) \text{ and } u \geq 1 \text{ in } K \right\}. \tag{5.1.1}$$

In the minimization problem (5.1.1), K is called the *obstacle*.

Variational capacities are for instance essential to study local behavior of solutions to quasilinear elliptic partial differential equations of second order. A pivotal result ([79], Thm. 10 or [62], Thm. 2.116 and 3.15) about *removable singularities* is that under relevant assumptions, a continuous solution (resp. a lower bounded weak solution) of such an equation, defined in a given domain $\Omega \subset \mathbb{R}^N$, except on a compact subset K, can be extended into a continuous solution (resp. a lower bounded weak solution) of the equation in Ω, provided that K has a null capacity. Therefore literature focuses on sets with null capacity, sometimes called 'polar sets' ([62], §2.1.2),

Another pivotal feature is the following nullity rule ([1] or [62], §2.1.7). Let $K \subset \mathbb{R}^N$ be a compact submanifold of dimension k. Then it holds:

$$c_p(K) = 0 \quad \text{if and only if} \quad p \leq N - k. \tag{5.1.2}$$

It follows that when $p > N$, any non-empty set has a positive variational capacity. Accordingly only the case $p \leq N$ is usually considered ([17], §5.8.2).

In contrast, we shall focus here on the fact that, choosing parameter p large enough according to rule (5.1.2), capacities take positive values on sets with zero Lebesgue measure, and in particular on compacts which codimensions are ≥ 2, such as points in 2D images or curves in 3D images.

The proof of rule (5.1.2) relies on comparison with Hausdorff measures ([62], §2.1.7 and [17], Thm. 4.1.2). Hausdorff measures have indeed been introduced extensively in functionals used in applicative fields, in particular in the field of imaging (e.g. [18]). Though it has proved to be a fruitful means, it appeared that introducing such measures also raises major challenges, for instance when it comes to quantify lengths of curves or to detect points in imaging (see e.g. [19] about the implementation of the Mumford-Shah functional [69] by means of Γ-convergence [37, 28] and more recently in [45] about the detection of points in a 2D-image).

This chapter is not meant at proposing finalized alternatives to the introduction of Hausdorff measures in practical applications. Its purpose is to provide answers on how to estimate capacities, when positive, and on how to obtain topological asymptotic expansions in the sense of expansion (1.1.1), when one geometric parameter of compact K_ε goes down to zero along with a parameter $\varepsilon > 0$.

We investigate in more detail two limit cases:

1. The first one deals with the approximation of the capacity of a point $\{x_0\}$. This first case admits sharp estimates by means of radial solutions obtained when obstacles are spherical compacts $\bar{B}(x_0, \varepsilon)$, $\varepsilon \to 0$.
2. In the second case, curves are in focus. A smooth curve can be locally approximated by a segment S_ε, provided that its length $\varepsilon > 0$ is small enough. Hence we study the capacity of a such segment S_ε. The effect caused by such an anisotropic obstacle to the p-Laplace equation has never been studied before. We give evidences of the strongly anisotropic effect caused by the segment and provide tools to estimate its capacity.

So as to be consistent with topological asymptotic expansions as defined in (1.1.1), we study a type of capacity which considers a compact set K within a given bounded domain Ω. Namely *condenser p-capacities* as defined in [49] will be considered all through this chapter.

5.1.2 Definition of condenser p-capacities

Since a Poincaré inequality holds in $W_0^{1,p}(\Omega)$ ([17],§5.3), Heinonen, Kilpeläinen and Martio [49] set

Definition 5.1.1. Let $W(K,\Omega) := \{v \in C_0^\infty(\Omega) : v \geq 1 \text{ in } K\}$. One defines

$$C_{p,N}(K,\Omega) := \inf_{v \in W(K,\Omega)} \int_\Omega |\nabla v|^p. \tag{5.1.3}$$

The nonnegative number $C_{p,N}(K,\Omega)$ is called the *p-capacity of the condenser* (K,Ω) or the *condenser p-capacity of the obstacle K in the bounded domain* Ω.

Using an approximation argument ([49], p. 27), one can prove that the set $W(K,\Omega)$ can be replaced in Definition 5.1.1 by the larger set

$$W_0(K,\Omega) := \left\{v \in W_0^{1,p}(\Omega) \cap C(\Omega) : v \geq 1 \text{ in } K\right\},$$

that is

$$C_{p,N}(K,\Omega) = \inf_{v \in W_0(K,\Omega)} \left\{ \int_\Omega |\nabla v|^p \right\}. \tag{5.1.4}$$

Again compact K is called the *obstacle* of the condenser. A function $v \in W_0(K,\Omega)$ is called an *admissible function* for the condenser.

Definition 5.1.1 can be extended to any subset $E \subset \Omega$, but we only need the case $E = K$ compact for estimation purposes in this chapter.

The concept of condenser capacity as defined by (5.1.3) differs from that of variational capacity in the sense of (5.1.1). As Ω is bounded, let $C_\Omega > 0$ be an admissible constant for the Poincaré inequality which holds in $W_0^{1,p}(\Omega)$. It is straightforward that

$$c_p(K) \leq (C_\Omega^p + 1)\ C_p(K,\Omega). \tag{5.1.5}$$

If K is such that $C_p(K,\Omega_0) > 0$ for a given bounded domain Ω_0, no reverse inequality holds uniformly in $\Omega \subset \Omega_0$. Indeed $C_p(K,\Omega) \to \infty$ when Ω decreases to K (for instance let $R \to \varepsilon$ in Proposition 5.2.2 hereafter).

For simplicity, we henceforth drop the word 'condenser' and simply say 'p-capacity' instead of 'condenser p-capacity' when no confusion is possible. Similarly we drop the 'N' of '$C_{p,N}(K,\Omega)$' simply writing '$C_p(K,\Omega)$' whenever no confusion is possible about the dimension of the ambient space.

Condenser capacities comply with Choquet's axiomatic definition, as after [49] §2.2, it holds

Theorem 5.1.2. *The set function $K \to C_p(K,\Omega)$, where K is a compact included in the domain $\Omega \subset \mathbb{R}^N$, enjoys the following properties:*

(i) (Monotony) If $K_1 \subset K_2 \subset \Omega$ then $C_p(K_1,\Omega) \leq C_p(K_2,\Omega)$.

(ii) (Monotony) If $K \subset \Omega_1 \subset \Omega_2$ then $C_p(K,\Omega_2) \leq C_p(K,\Omega_1)$.

(iii) (Subadditivity) If $K_1 \subset \Omega$ and $K_2 \subset \Omega$ then

$$C_p(K_1 \cup K_2,\Omega) + C_p(K_1 \cap K_2,\Omega)\ \leq\ C_p(K_1,\Omega) + C_p(K_2,\Omega).$$

(iv) (Descending continuity) If $(K_n)_{n\geq 0}$ is a decreasing sequence of compact subsets of Ω, that is $\Omega \supset K_0 \supset K_1 \supset \cdots \supset K_n \supset K_{n+1} \supset \cdots$ and $K := \bigcap_{n\geq 0} K_n$, then

$$C_p(K,\Omega) = \lim_{n\to+\infty}\ C_p(K_n,\Omega).$$

(v) (Ascending continuity) If $(K_n)_{n\geq 0}$ is an ascending sequence of compact subsets of Ω and if $K := \bigcup_{n\geq 0} K_n$ is compact, then

$$C_p(K,\Omega) = \lim_{n\to+\infty} C_p(K_n,\Omega).$$

(vi) If $(K_n)_{n\geq 0}$ is a sequence of compact subsets of Ω and if $K := \bigcup_{n\geq 0} K_n$ is compact, then

$$C_p(K,\Omega)\ \leq\ \sum_{n=0}^{+\infty} C_p(K_n,\Omega).$$

The two monotony properties (i) and (ii) and the descending continuity property (iv) will often be applied hereafter.

5.1.3 State of the art

Our goal is to study estimates and when possible, asymptotic expansions of p-capacities of condensers, for obstacles with non-empty interior as well as for points and for segments.

Most available estimates of capacities deal with the *harmonic* or *electrostatic* case, that is $p = 2$, either referring to variational capacities (e.g. [43, 85]) or to condenser capacities (e.g. [73, 74]).

Results usually come in the form of inequalities. Equalities are exceptions. When $p \neq 2$ actually, as mentioned by [49] §2.11, only were calculated p-capacities of spherical condensers.

One *a priori* expects the condenser p-capacity of compact K in Ω to depend on the shape of K but also on its localization within Ω and on the shape and size of Ω.

While the condenser capacity of a point $\{x_0\}$ may be approximated by condenser capacities of spherical compacts $\bar{B}(x_0, \varepsilon)$, $\varepsilon \to 0$, the estimation of condenser capacities of segments remains an opened question. Trying to apply the descending continuity property of capacities, one may estimate capacities of ellipsoids. Some results are available for ellipsoids (e.g. [73, 85, 43]) but only in the harmonic case $p = 2$. With respect to p-Laplace problems, most available results address the case of isolated singularities and of radial solutions ([54, 90]). Anisotropic solutions of the form $u(x) = |x|^\lambda \ \omega(x/|x|)$, where $\lambda \in \mathbb{R}$ and ω is defined on the unit sphere S^{N-1}, were studied for quasilinear equations with Dirichlet conditions in domains with conical boundary points [86, 75]. To our best knowledge, the anisotropic effect caused by a segment in the p-Laplace equation has not been studied yet.

Lastly, with respect to positivity versus nullity cases of condenser p-capacities, clearly if the variational p-capacity of a compact K in $\mathbb{R}^N$ is positive, then the condenser p-capacity of K in a bounded domain is positive. The necessary and sufficient condition of nullity for the variational p-capacity of a set is well-known (see e.g. [62], §2.1.7 and [17], Thm. 4.1.2), i.e. parameter p has to be less or equal the codimension of the set. But available results ([49], Theorems 2.26 and 2.27) about cases of nullity for condenser p-capacities only apply when the condenser p-capacity of a compact K is null *in all bounded domain* Ω, $K \subset \Omega \subset \mathbb{R}^N$.

5.1.4 Overview of chapter 5

Section 5.2 is devoted to definitions and preliminary estimation tools. To estimate the capacity of an obstacle with empty interior, due to the descending continuity property of capacities, it matters to estimate condenser p-capacities of decreasing sequences of obstacles K_ε with non-empty interior, such that $\bigcap\limits_{\varepsilon>0} K_\varepsilon$ reduces to the targeted obstacle. We show that one can calculate a condenser p-capacity by solving a p-Laplace equation with Dirichlet boundary condition, when the boundaries of the condenser are smooth. Then we give estimates for condenser p-capacities when the obstacle K has a non-empty interior.

In section 5.3, we apply the previous results to obtain asymptotic approximations and cases of positivity for condenser p-capacities of points. We study the convergence speed of descending continuity.

Our main contributions deal with condenser p-capacities of segments in section 5.4. For this purpose, we first introduce *equidistant condensers*. As a first illustration of the strong anisotropy of the problem, we show that the Pólya-Szegö rearrangement inequality for Dirichlet type integrals fails to provide a valuable lower bound to the p-capacity of a segment. As a second illustration, when $p > N$, we show that one cannot simply derive an admissible solution for the segment S_ε, however small its length $\varepsilon > 0$ may be, from the case of a punctual obstacle.

Taking into account the outcome of the previous experiments, our main contribution is to provide a lower bound to the condenser p-capacity of a segment S in a N-dimensional bounded domain Ω, by means of the N-dimensional p-capacity of a point and more importantly by means of the $(N-1)$-dimensional p-capacity of a point. We can then obtain directly positivity cases for condenser p-capacities of segments in an N-dimensional bounded domain Ω. This method could be extended to obstacles of higher dimensions, e.g. for plane rectangles in $\mathbb{R}^N$, $N \geq 4$.

In section 5.4.6 we introduce *elliptical condensers*, defined in elliptic coordinates. The angular coordinate ν so to speak makes the dimension in which operates the p-Laplace equation, continuously change from N for $\nu = 0$ to $(N-1)$ for $\nu = \pi/2$ and then back to N for $\nu = \pi$. We provide an estimate for the condenser 2-capacity of a segment in the plane and its asymptotic expansion when the segment length goes down to 0 (in higher dimensions, this 2-capacity is null). In terms of topological asymptotic expansions, this result shows that the 2-condenser capacity in the plane is unable to separate curves and balls. When $p \neq 2$, elliptical condensers could prove useful to obtain further estimations of condenser p-capacities of segments.

For reader's convenience, two proofs requiring longer calculations are postponed to section 5.5.

5.2 Preliminary results for condenser capacities

5.2.1 Estimating p-capacity by means of a p-Laplace problem

Assume that both Ω and K have smooth C^1-boundaries. Consider the following p-Laplace problem in $\Omega \setminus K$ with Dirichlet boundary condition:

$$\begin{cases} -\Delta_p(u) = 0 & \text{in } \Omega \setminus K \\ u = 1 & \text{on } \partial K \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2.1}$$

where Δ_p denotes the p-Laplace operator $\Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

We recall some well-known facts about Problem (5.2.1) (see e.g. [60], Thm. 2.16 or [17], §6.6 for existence and uniqueness; [87, 91, 46, 62] for regularity properties; and [60, 46, 62] for the Maximum Principle):

- Problem (5.2.1) admits a unique solution $u \in W^{1,p}(\Omega \setminus K)$.
- One can define solution u as being the unique function that minimizes the Fréchet differentiable, strictly convex and coercive functional

$$J : v \in W^{1,p}(\Omega \setminus K) \mapsto \int_{\Omega\setminus K} |\nabla v|^p \tag{5.2.2}$$

in the affine space $g + W_0^{1,p}(\Omega \setminus K)$, where $g \in C_0^\infty(\Omega)$ is chosen such that $g = 1$ on a neighborhood Ω' of K, with $K \subset\subset \Omega' \subset\subset \Omega$. In other words,

$$\{u\} = \underset{v \in g + W_0^{1,p}(\Omega\setminus K)}{\operatorname{argmin}} \int_{\Omega\setminus K} |\nabla v|^p. \tag{5.2.3}$$

- Function u is continuous in $\overline{\Omega \setminus K}$ (after a redefinition in a set of zero measure) and u is C^1 in $\Omega \setminus K$. In particular it holds $u = 0$ on $\partial\Omega$ and $u = 1$ on ∂K pointwise.
- It holds

$$0 < u(x) < 1, \quad \forall x \in \Omega \setminus K. \tag{5.2.4}$$

Proposition 5.2.1. *Let K be a compact set of a bounded domain $\Omega \subset \mathbb{R}^N$, both with C^1-boundaries. Let $u \in W^{1,p}(\Omega \setminus K) \cap C\left(\overline{\Omega \setminus K}\right)$ be the unique solution to Problem (5.2.1). Then*

$$C_p(K, \Omega) = \int_{\Omega\setminus K} |\nabla u|^p. \tag{5.2.5}$$

Moreover let $\tilde{u}$ be the extension of u in Ω obtained by setting $\tilde{u} = 1$ in K. Then $\tilde{u} \in W_0(K,\Omega)$ and $\tilde{u}$ minimizes Problem (5.1.4), that is

$$\{\tilde{u}\} = \underset{v \in W_0(K,\Omega)}{\operatorname{argmin}} \int_\Omega |\nabla v|^p. \tag{5.2.6}$$

and

$$C_p(K, \Omega) = \int_\Omega |\nabla \tilde{u}|^p.$$

Proof. The regularity of function u entails that $\tilde{u} \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$. As $\tilde{u} = 1$ in K, it follows that $\tilde{u} \in W_0(K,\Omega)$. Obviously $\nabla\tilde{u} = 0$ in $\mathring{K}$. Hence according to (5.1.4)

$$C_p(K, \Omega) \leq \int_\Omega |\nabla \tilde{u}|^p = \int_{\Omega\setminus K} |\nabla u|^p. \tag{5.2.7}$$

Conversely, according to definition (5.1.1), let $(u_n)_{n\geq 0}$ a sequence $\subset W(K,\Omega)$ such that

$$C_p(K, \Omega) = \lim_{n\to+\infty} \int_\Omega |\nabla u_n|^p.$$

For all $n \geq 0$, define $w_n := \inf(u_n, 1)$ in Ω. It follows from [49] Thm 1.20 that

$$w_n \in W_0^{1,p}(\Omega) \cap C(\Omega)$$

and that

$$|\nabla w_n(x)| \leq |\nabla u_n(x)|, \quad \text{for a.e. } x \in \Omega.$$

In addition the obstacle condition $u_n \geq 1$ in K implies that $w_n = 1$ in K and $\nabla w_n = 0$ in $\mathring{K}$. Hence $w_n \in W_0(K,\Omega)$ with

$$\int_\Omega |\nabla w_n|^p \leq \int_\Omega |\nabla u_n|^p.$$

Let v_n be the restriction of w_n to $\Omega \setminus K$. We check that $v_n - g \in W_0^{1,p}(\Omega \setminus K)$ where function g was defined in (5.2.3). Therefore according to (5.2.3) for all $n \geq 0$ it holds:

$$J(u) = \int_{\Omega\setminus K} |\nabla u|^p \leq \int_{\Omega\setminus K} |\nabla v_n|^p = \int_{\Omega} |\nabla w_n|^p \leq \int_{\Omega} |\nabla u_n|^p.$$

Letting $n \to +\infty$ yields

$$\int_{\Omega\setminus K} |\nabla u|^p \leq C_p(K, \Omega).$$

Comparing with (5.2.7) one concludes

$$C_p(K, \Omega) = \int_{\Omega} |\nabla \tilde{u}|^p = \int_{\Omega\setminus K} |\nabla u|^p. \tag{5.2.8}$$

Equality (5.2.5) is thus proved. As we already noticed that $\tilde{u} \in W_0(K, \Omega)$, it also follows that

$$\{\tilde{u}\} = \operatorname*{argmin}_{v\in W_0(K,\Omega)} \left\{ \int_{\Omega} |\nabla v|^p \right\}, \tag{5.2.9}$$

which completes the proof. □

Note that after (5.2.4), obviously

$$0 < \tilde{u}(x) < 1, \quad \forall x \in \Omega \setminus K. \tag{5.2.10}$$

Thus after Proposition 5.2.1 one can estimate the capacity of the condenser (K, Ω) by estimating the energy of the solution to Problem (5.2.1) when boundaries ∂K and $\partial\Omega$ are smooth enough.

If boundaries ∂K or ∂K are not C^1, then thanks to the two monotony properties (i) and (ii) of Theorem 5.1.2, one can estimate $C_p(K, \Omega)$ as long as K (resp. Ω) can be properly approximated respectively by (a sequence of) some other compact sets (resp. open sets) with C^1-boundaries to which one may in turn apply Proposition 5.2.1. This approximation technique will be applied in subsection 5.2.3 hereafter.

It is convenient to extend the definition of 'admissible function' for a condenser, given in Definition 5.1.1, as follows : let $v \in W^{1,p}(\Omega \setminus K) \cap C(\overline{\Omega \setminus K})$ such that $v = 0$ on $\partial\Omega$ and $v = 1$ on ∂K. Let $\tilde{v}$ be the extension of v in Ω obtained by setting $\tilde{v} = 1$ in K. Clearly $\tilde{v}$ is admissible for the condenser (K, Ω) in the sense of Definition 5.1.1. By extension we thus say that function v is admissible for the condenser (K, Ω).

5.2.2 Asymptotic expansions of capacity for spherical condensers

Let a point $x_0 \in \mathbb{R}^N$, two numbers $0 < \varepsilon < R$ and the concentric balls $\overline{B}_\varepsilon := \overline{B}(x_0, \varepsilon)$ and $B_R := B(x_0, R)$. For simplicity, we denote $C_p(\varepsilon, R)$ the p-capacity of the spherical condenser $\left(\overline{B}(x_0, \varepsilon), B(x_0, R)\right)$. We recall and detail hereafter the well-known result (e.g. [49], §2.11) about the spherical condenser $(\bar{B}_\varepsilon, B_R)$ as shown on Figure 5.1.

Proposition 5.2.2. *Denote $s_{p,N} \in W^{1,p}(B_R \setminus \overline{B}_\varepsilon)$ the unique solution to Problem (5.2.1) when $K = \overline{B}_\varepsilon$ and $\Omega = B_R$. Denote $r = |x - x_0|$ for all $x \in B_R \setminus \overline{B}_\varepsilon$.*

1. *If $p = N$, then for all $x \in B_R \setminus \overline{B}_\varepsilon$ it holds:*

$$\begin{cases} s_{p,N}(x) = \left[\log(R/r)/\log(R/\varepsilon)\right], \\ |\nabla s_{p,N}(x)| = \left[r\log(R/\varepsilon)\right]^{-1}, \end{cases}$$

$$C_p(\varepsilon, R) = A^{N-1}\left[\log(R/\varepsilon)\right]^{1-p},$$

and for $\varepsilon > 0$ small enough:

$$C_p(\varepsilon, R) = A^{N-1}\left[-\log\varepsilon\right]^{1-p}\left[1 + (p-1)(\log R/\log\varepsilon) + o\left(1/\log\varepsilon\right)\right].$$

2. *If $p \neq N$, then for all $x \in B_R \setminus \overline{B}_\varepsilon$ it holds:*

$$\begin{cases} s_{p,N}(x) = (R^\beta - r^\beta)/(R^\beta - \varepsilon^\beta), \\ |\nabla s_{p,N}(x)| = \left|\frac{\beta}{R^\beta - \varepsilon^\beta}\right| r^{\beta-1}, \end{cases}$$

$$C_p(\varepsilon, R) = A^{N-1}|\beta|^{p-1}\left|R^\beta - \varepsilon^\beta\right|^{1-p},$$

and for $\varepsilon > 0$ small enough

$$\begin{cases} C_p(\varepsilon, R) = A^{N-1}\beta^{p-1}R^{N-p}\left[1 + (p-1)\left(\varepsilon/R\right)^\beta + o\left(\varepsilon^\beta\right)\right], & \text{if } p > N, \\ C_p(\varepsilon, R) = A^{N-1}\left(-\beta\right)^{p-1}\varepsilon^{N-p}\left[1 + (p-1)\left(\varepsilon/R\right)^{-\beta} + o\left(\varepsilon^{-\beta}\right)\right], & \text{if } p < N. \end{cases}$$

A proof is available in subsection 5.5.1 on page 134. It is obtained solving Problem (5.2.1) in spherical coordinates and then applying Proposition 5.2.1.

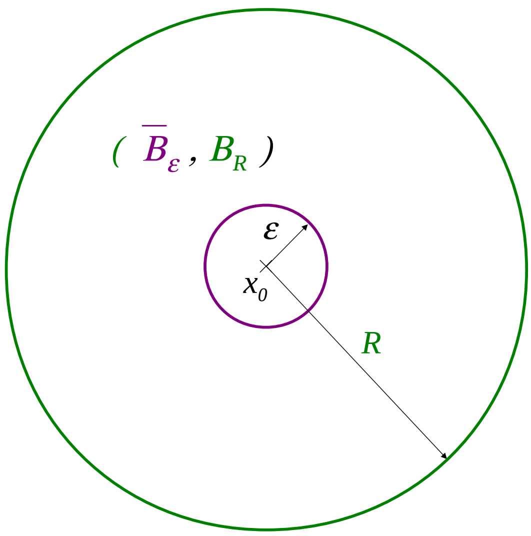

Figure 5.1: A spherical condenser $(\overline{B}_\varepsilon, B_R)$.

5.2.3 Capacities of condensers which obstacle has non-empty interior

Let a point $x_0 \in \Omega$. Let the two numbers

$$R_1 := \sup\{R > 0; B(x_0, R) \subset \Omega\} > 0 \quad \text{and} \quad R_2 := \inf\{R > 0; \Omega \subset B(x_0, R)\}.$$

Let a non-empty bounded domain $\omega \subset \mathbb{R}^N$ such that $0 \in \omega$. Let the two numbers

$$\rho_1 := \sup\{\rho > 0; B(x_0, \rho) \subset \omega\} \quad \text{and} \quad \rho_2 := \inf\{\rho > 0; \omega \subset B(x_0, \rho)\}.$$

Let $\omega_\varepsilon := x_0 + \varepsilon \cdot \omega \subset B(x_0, R_1)$ for $\varepsilon > 0$ small enough and consider the condenser $(\overline{\omega}_\varepsilon, \Omega)$ as shown on Figure 5.2.

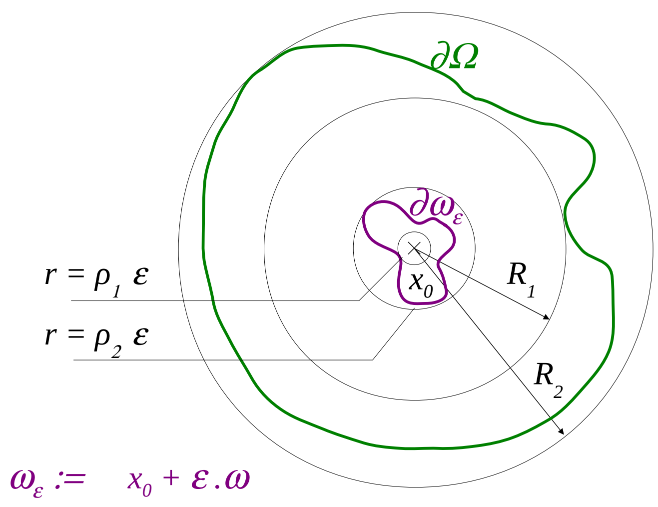

Figure 5.2: A condenser which obstacle $\overline{\omega}_\varepsilon$ has a non-empty interior.

Proposition 5.2.3. *The following asymptotic inequalities hold.*

1. *If $p = N$, then:*

$$-A^{N-1}(p-1)\log(R_2/\rho_1)\,[-\log\varepsilon]^{-p} + o\left([\log\varepsilon]^{-p}\right)$$

$$\leq \quad C_p(\overline{\omega}_\varepsilon, \Omega) - A^{N-1}\,[-\log\varepsilon]^{1-p} \quad \leq$$

$$-A^{N-1}(p-1)\log(R_1/\rho_2)\,[-\log\varepsilon]^{-p} + o\left([\log\varepsilon]^{-p}\right).$$

2. *If $p > N$, then:*

$$A^{N-1}\beta^{p-1}R_2^{N-p}\left[1 + (p-1)\,(\rho_1\,\varepsilon/R_2)^\beta + o\left(\varepsilon^\beta\right)\right]$$

$$\leq \quad C_p(\overline{\omega}_\varepsilon, \Omega) \quad \leq$$

$$A^{N-1}\beta^{p-1}R_1^{N-p}\left[1 + (p-1)\,(\rho_2\,\varepsilon/R_1)^\beta + o\left(\varepsilon^\beta\right)\right].$$

3. *If $p < N$, then:*

$$A^{N-1}\,(-\beta)^{p-1}\,(\rho_1\,\varepsilon)^{N-p}\left[1+(p-1)\,(\rho_1\,\varepsilon/R_2)^{-\beta}+o\left(\varepsilon^{-\beta}\right)\right]$$

$$\leq \quad C_p(\overline{\omega}_\varepsilon,\Omega) \quad \leq$$

$$A^{N-1}\,(-\beta)^{p-1}\,(\rho_2\,\varepsilon)^{N-p}\left[1+(p-1)\,(\rho_2\,\varepsilon/R_1)^{-\beta}+o\left(\varepsilon^{-\beta}\right)\right].$$

Proof. Let four positive real numbers ρ', ρ'', R' and R'' such that

$$B(x_0,\rho'\varepsilon)\subset\omega_\varepsilon\subset B(x_0,\rho''\varepsilon)\subset B(x_0,R')\subset\Omega\subset B(x_0,R'').$$

According to monotony properties (i) and (ii) of Theorem 5.1.2, the following inequalities hold

$$C_p(\rho'\varepsilon,R'')\leq C_p(\overline{B}_{\rho'\varepsilon},\Omega)\leq C_p(\overline{\omega}_\varepsilon,\Omega)\leq C_p(\overline{B}_{\rho''\varepsilon},\Omega)\leq C_p(\rho''\varepsilon,R'). \tag{5.2.11}$$

The formula of $C_p(\rho,R)$ provided by Theorem 5.2.2 shows that the map $(\rho,R)\in\mathbb{R}^2\mapsto C_p(\rho,R)$ is continuous in $\{(\rho,R)\in\mathbb{R}^2 \ | \ 0<\rho<R\}$.

Hence letting $R'\nearrow R_1$, $R''\searrow R_2$, $\rho'\nearrow\rho_1$ and $\rho''\searrow\rho_2$ in inequalities (5.2.11), it follows that

$$C_p(\rho_1\varepsilon,R_2)\leq C_p(\overline{\omega}_\varepsilon,\Omega)\leq C_p(\rho_2\varepsilon,R_1).$$

Then consider the asymptotic expansions provided by Theorem 5.2.2 for the lower-bound $C_p(\rho_1\varepsilon,R_2)$ and for the upper-bound $C_p(\rho_2\varepsilon,R_1)$. The expansions claimed in Proposition 5.2.3 straightforwardly follow, whether $p=N$, $p>N$ or $p<N$. □

Note that no assumption is required about the smoothness of boundaries $\partial\omega$ and $\partial\Omega$.

Remark 5.2.4. The expansions stated in Proposition 5.2.3 are actually topological asymptotic expansions in the sense of 1.1.1.

1. If $p=N$, then the expansion reads

$$C_p(\overline{\omega}_\varepsilon,\Omega)=A^{N-1}\,[-\log\varepsilon]^{1-p}+o\left([-\log\varepsilon]^{1-p}\right).$$

   The topological gradient equals A^{N-1}. It is constant in Ω. It does not depend on the shape of the compact $\overline{\omega}$ nor on that of the domain Ω.

2. If $p<N$ and if ω is the unit ball, then the expansion reads

$$C_p(\overline{B}_\varepsilon,\Omega)=A^{N-1}\,(-\beta)^{p-1}\,\varepsilon^{N-p}+o\left(\varepsilon^{N-p}\right).$$

   The topological gradient equals $A^{N-1}\,(-\beta)^{p-1}$. It is constant in Ω. It does not depend on the shape of the domain Ω.

3. In the harmonic case $p=2$ and for $N=2$ or $N=3$, the results here above comply with the topological asymptotic expansion previously proved in [47] for the Laplace equation with Dirichlet boundary condition.

Remark 5.2.5. According to expansions stated in Proposition 5.2.3, the domain Ω, through parameters R_1 and R_2, does not impact the main term of the asymptotic expansion of the capacity $C_p(\overline{\omega}_\varepsilon, \Omega)$ when $p \leq N$.

In contrast when $p > N$, the localization of x_0 within Ω and the shape of Ω determine the main term of the expansion through parameters R_1 and R_2. This case exemplifies a major difference between the concept of condenser capacities in a bounded domain Ω and that of variational capacities in $\mathbb{R}^N$.

5.3 Condenser p-capacity of a point and approximations

5.3.1 Condenser p-capacity of a point

Proposition 5.3.1. *Let x_0 be a point of a bounded domain $\Omega \subset \mathbb{R}^N$. The following positivity rule holds:*

$$C_p(\{x_0\}, \Omega) > 0 \quad \textit{if and only if} \quad p > N. \tag{5.3.1}$$

Moreover, if $p > N$, then:

$$A^{N-1}\beta^{p-1}R_2^{N-p} \leq C_p(\{x_0\}, \Omega) \leq A^{N-1}\beta^{p-1}R_1^{N-p} \tag{5.3.2}$$

where $R_1 := \sup\{R > 0; B(x_0, R) \subset \Omega\}$ and $R_2 := \inf\{R > 0; \Omega \subset B(x_0, R)\}$.
In particular, if $p > N$ and if $\Omega = B(x_0, R)$, then it holds

$$C_p(\{x_0\}, B_R) = A^{N-1}\beta^{p-1}R^{N-p}.$$

Proof. As $x_0 \in \Omega$, there exists $\delta > 0$ such that $B(x_0, \delta) \subset \Omega$. Let $\omega := B(0, \delta)$ and $\omega_\varepsilon := x_0 + \varepsilon\omega$ for all $\varepsilon > 0$.

For all $\varepsilon \in (0, 1)$, the estimates of $C_p(\overline{\omega}_\varepsilon, \Omega)$ stated in Proposition 5.2.3 hold. As

$$\{x_0\} = \bigcap_{\varepsilon \in (0,1)} \overline{\omega}_\varepsilon,$$

it follows from the descending continuity property of Theorem 5.1.2 that

$$\lim_{\varepsilon \to 0} C_p(\overline{\omega}_\varepsilon, \Omega) = C_p(\{x_0\}, \Omega).$$

Passing to the limit when $\varepsilon \to 0$ in estimates of $C_p(\overline{\omega}_\varepsilon, \Omega)$ provided by Proposition 5.2.3, the claimed results follow whether $p = N$, $p > N$ or $p < N$. $\square$

5.3.2 Speed of convergence of descending continuity

According to the descending property of Theorem 5.1.2, one can approximate the capacity of an obstacle with zero measure by calculating capacities of obstacles with positive measures going down to zero. From this perspective, the speed of convergence of descending continuity becomes a point of interest.

This question can be answered in the case of a point. If one wishes to obtain an estimate of the capacity of a point, one can calculate the capacity of a ball with a small enough r. How small should this radius be, depending on the maximum acceptable error for the value of the capacity of the point?

Proposition 5.3.2. *If $p > N$, for $0 < r < R$, it holds*

$$C_p(\overline{B}(x_0, r), B(x_0, R)) - C_p(\{x_0\}, B(x_0, R)) = O\left(r^\beta\right) \tag{5.3.3}$$

Proof. The claimed estimate follows straightforwardly from the expansion stated in Proposition 5.2.2 in the case $p > N$ and from the value of $C_p(\{x_0\}, B(x_0, R))$ provided by Proposition 5.3.1. □

When $p > N \geq 2$, it holds $0 < \beta < 1$. Unfortunately, the speed of convergence to zero, that is $O\left(r^\beta\right)$, is slow when $r \to 0$.

Moreover according to Proposition 5.2.2, for all $\varepsilon \in (0, R)$ it holds

$$C_p(\varepsilon, R) = A^{N-1}\beta^{p-1}\left(R^\beta - \varepsilon^\beta\right)^{1-p},$$

and thus the derivative

$$\frac{dC_p(\varepsilon, R)}{d\varepsilon} = A^{N-1}\beta^p(p-1)\left(R^\beta - \varepsilon^\beta\right)^{-p}\varepsilon^{\beta-1}.$$

Hence

$$\lim_{\varepsilon\to 0}\frac{dC_p(\varepsilon, R)}{d\varepsilon} = +\infty,$$

which confirms that the speed of descending continuity is slow.

5.4 Estimates of p-capacities of segments

In this section 5.4, a segment will be a compact set $S_\varepsilon \subset \mathbb{R}^N$, $N \geq 2$, defined as follows

$$S_\varepsilon := \{x_0 + z\tau\,;\, z \in [-\varepsilon/2, \varepsilon/2]\}$$

where x_0 is the center of the segment, $\varepsilon > 0$ its length and $\tau \in \mathbb{R}^N$ is a unit vector.

As recalled in section 5.1.3, the effect of the anisotropy caused by a segment acting as an obstacle in the p-Laplace equation remains unknown.

5.4.1 Equidistant condensers

Recall K is a compact subset of the bounded domain $\Omega \subset \mathbb{R}^N$. For all $x \in \mathbb{R}^N$ and all subset $E \subset \mathbb{R}^N$, we denote the distance $d(x, E) = \inf\{|x' - x|\,;\, x' \in E\}$

Definition 5.4.1. Let $0 < \eta < R$. Let the compact $K_\eta := \left\{x \in \mathbb{R}^N \mid d(x, K) \leq \eta\right\}$ and the bounded domain $\Omega_R := \left\{x \in \mathbb{R}^N \mid d(x, K) < R\right\}$. We say that (K_η, Ω_R) is an *equidistant condenser derived from the compact K.*

Let $0 < \eta < R$ and consider the *equidistant condenser* (K_η, Ω_R) *derived from the segment* S_ε as shown on Figure 5.3.

Some notations and remarks are useful.

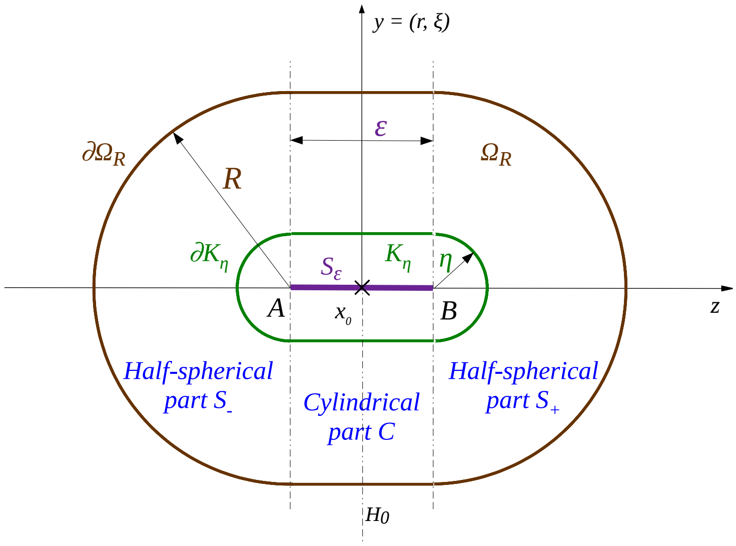

Figure 5.3: An equidistant condenser (K_η, Ω_R).

1. Let z be an axis passing through the point x_0 and parallel to the segment S_ε. Due to the symmetry of revolution of the condenser (K_η, Ω_R) around the z-axis, it is convenient to use the cylindrical coordinates $x = (z, y) = (z, r, \xi)$, with $z \in \mathbb{R}$, $y = r\xi \in \mathbb{R}^{N-1}$, $r \geq 0$ and $\xi \in S^{N-2}$. Let A (resp. B) the point of cylindrical coordinates $(z = -\varepsilon/2, r = 0)$ (resp. $(z = \varepsilon/2, r = 0)$).
   Let
   $$C := \{x \in \Omega_R \setminus K_\eta \,;\, |z| < \varepsilon/2\}$$
   be the open cylindrical subset of $\Omega_R \setminus K_\eta$ and
   $$S_\pm := \{x \in \Omega_R \setminus K_\eta \,;\, \pm z > \varepsilon/2\}$$
   the two open half-spherical subsets of $\Omega_R \setminus K_\eta$.
   Denote $S := S_- \cup S_+$. In particular $(\Omega_R \setminus K_\eta) \setminus (C \cup S)$ is of zero Lebesgue measure.

2. As in subsection 5.2.1, we denote $u \in W^{1,p}(\Omega_R \setminus K_\eta) \cap C\left(\overline{\Omega_R \setminus K_\eta}\right) \cap C^1\left(\Omega_R \setminus K_\eta\right)$ the unique solution to Problem (5.2.1) when $K = K_\eta$ and $\Omega = \Omega_R$. According to Proposition 5.2.1, the p-capacity of the condenser (K_η, Ω_R) is given by
   $$C_{p,N}\left(K_\eta, \Omega_R\right) = \int_{C \cup S} |\nabla u|^p \, dx$$
   where ∇ denotes the gradient operator in $\mathbb{R}^N$ and dx the Lebesgue measure in $\mathbb{R}^N$.
   Moreover S_ε is invariant by the orthogonal symmetry $(z, y) \mapsto (-z, y)$ relative to $H_0 := \{z = 0\}$. Thus the condenser (K_η, Ω_R) enjoys the same symmetry and so does u due to uniqueness of the solution to Problem (5.2.1).

3. Denote $s_{p,N}$ the admissible function minimizing the energy of the N-dimensional spherical condenser $(\overline{B}(x_0,\eta), B(x_0,R))$. After Proposition 5.2.1 it holds
$$C_{p,N}(\eta,R) = \int_{B(x_0,R)\setminus B(x_0,\eta)} |\nabla s_{p,N}|^p \, dx.$$
The values of $s_{p,N}$ and $C_{p,N}(\eta,R)$ were provided in Proposition 5.2.2.
4. For all $a \in [-\varepsilon/2, \varepsilon/2]$, let H_a be the affine hyperplane $\{z = a\}$ and $\{x_a\}$ the intersection between H_a and the z-axis.
It is pivotal to note that $(K_\eta \cap H_a, \Omega_R \cap H_a)$ is a $(N-1)$-dimensional spherical condenser. The admissible function minimizing the energy of this condenser is denoted $s_{p,N-1}$. Similarly after Proposition 5.2.1, it holds
$$C_{p,N-1}(\eta,R) = \int_{B_{N-1}(x_a,R)\setminus B_{N-1}(x_a,\eta)} |\nabla_y \; s_{p,N-1}|^p \, dy,$$
where B_{N-1} denotes a $(N-1)$-dimensional ball, ∇_y is the gradient operator in $\mathbb{R}^{N-1}$ and dy the Lebesgue measure in $\mathbb{R}^{N-1}$. The values of $s_{p,N-1}$ and $C_{p,N-1}(\eta,R)$ were provided in Proposition 5.2.2.

5.4.2 Pólya-Szegö rearrangement inequality for Dirichlet type integrals

While the definition 5.1.1 of a condenser p-capacity allows to obtain upper bounds by calculating energies of admissible functions, obtaining a lower bound to a capacity is a more difficult task. In [73, 74] G. Pólya and G. Szegö showed in the harmonic case $p = 2$, that the so called *Schwarz symmetrization* can provide a lower bound to a condenser 2-capacity. More recently Brothers and Ziemer [32, 42, 33] extended this method, known as the Pólya-Szegö rearrangement inequality for Dirichlet type integrals $\int_\Omega |\nabla u|^p$, for all $p \in [1,\infty)$.

So let us apply the Pólya-Szegö rearrangement inequality for Dirichlet type integrals, to obtain a lower-bound to the p-capacity of a segment S_ε.

– For $0 < \eta < R$, let the equidistant condenser (K_η, Ω_R). According to definition 5.1.1 and Proposition 5.2.1, let
$$u \in W^{1,p}(\Omega_R \setminus K_\eta) \cap C(\overline{\Omega_R \setminus K_\eta}) \cap C^1(\Omega_R \setminus K_\eta)$$
the solution of Problem (5.2.1). Let $\tilde{u} \in W_0^{1,p}(\Omega_R)$ the extension of u obtained by setting $\tilde{u} = 1$ in K_η. Recall
$$C_p(K_\eta, \Omega_R) = \int_{\Omega_R\setminus K_\eta} |\nabla u|^p = \|\nabla \tilde{u}\|^p_{L^p(\Omega_R)}$$
and
$$0 < \tilde{u}(x) < 1, \quad \forall x \in \Omega_R \setminus K_\eta.$$
– Let $\Omega^\sharp$ the open ball of $\mathbb{R}^N$ centered at the origin 0 such that $\left|\Omega^\sharp\right| = |\Omega_R|$. The radius $R^\sharp$ of $\Omega^\sharp$ is given by
$$R^\sharp := \left[R^{N-1}\left(R + \frac{A^{N-2}}{A^{N-1}}\,\varepsilon\right)\right]^{\frac{1}{N}}$$

and it holds $R^\sharp > R$.
Similarly we denote $K^\sharp$ the closed ball of $\mathbb{R}^N$ centered at the origin 0 such that $\left|K^\sharp\right| = |K_\eta|$. The radius $\eta^\sharp$ of $K^\sharp$ is given by

$$\eta^\sharp := \left[\eta^{N-1}\left(\eta + \frac{A^{N-2}}{A^{N-1}}\,\varepsilon\right)\right]^{\frac{1}{N}}.$$

– Let $\mu_{\tilde{u}}$ the distribution function of $\tilde{u}$ defined by

$$\mu_{\tilde{u}}(t) := \left|\{x \in \Omega_R; |\tilde{u}(x)| > t\}\right|, \quad \forall t \geq 0.$$

It holds $\mu_{\tilde{u}}(0) = |\Omega_R|$ and $\lim\limits_{t\to 1, t<1} \mu_{\tilde{u}}(t) = |K_\eta|$.

– Let u^* the non-increasing rearrangement of $\tilde{u}$ defined by

$$u^*(s) := \inf\{t \geq 0; \mu_{\tilde{u}}(t) \leq s\}, \quad \forall s \in (0, |\Omega_R|].$$

It holds $u^*(|K_\eta|) = 1$ and $u^*(|\Omega_R|) = 0$.

– Let $u^\sharp$ the symmetric rearrangement of $\tilde{u}$ defined by

$$u^\sharp(x) := u^*(A^{N-1}|x|^N), \quad \forall x \in \Omega^\sharp.$$

It holds

$$u^\sharp(x) = 1,\ \forall x \in \partial K_{\eta^\sharp} \ \text{ and } \ u^\sharp(x) = 0,\ \forall x \in \partial\Omega_{R^\sharp}.$$

The Pólya-Szegö rearrangement inequality for Dirichlet type integrals reads

$$\left\|\nabla u^\sharp\right\|^p_{L^p(\Omega^\sharp)} \leq \|\nabla\tilde{u}\|^p_{L^p(\Omega_R)}.$$

As $u^\sharp$ is an admissible function for the spherical condenser $(K_{\eta^\sharp}, \Omega_{R^\sharp})$, it follows

$$C_p(K_{\eta^\sharp}, \Omega_{R^\sharp}) \leq \left\|\nabla u^\sharp\right\|^p_{L^p(\Omega^\sharp)}.$$

Therefore

$$C_p(K_{\eta^\sharp}, \Omega_{R^\sharp}) \leq C_p(K_\eta, \Omega_R). \tag{5.4.1}$$

– As $\lim\limits_{\eta\to 0} \eta^\sharp = 0$, it holds $\bigcap\limits_{\eta>0} K_{\eta^\sharp} = \{0\}$. Hence applying the descending continuity property of Theorem 5.1.2 to inequality (5.4.1), it follows

$$C_p(\{0\}, B_{R^\sharp}) \leq C_p(S_\varepsilon, \Omega_R). \tag{5.4.2}$$

According to positivity rule for condenser p-capacity of points stated in Proposition 5.3.1 and to monotony properties stated in Theorem 5.1.2, inequality (5.4.2) provides no additional information:

1. If $p \leq N$, the lower bound $C_p(\{0\}, B_{R^\sharp})$ is null.
2. In the case $p > N$, one already knew that

$$C_p(\{0\}, B_{R^\sharp}) \leq C_p(\{0\}, B_R) \leq C_p(S_\varepsilon, B(x_0, R)) \leq C_p(S_\varepsilon, \Omega_R).$$

Hence the anisotropy caused by the segment in the p-Laplace problem is not appropriately estimated by the symmetric rearrangement method applied here above.

5.4.3 From the point to the segment ?

With notations of subsection 5.4.1, let us try to build an admissible solution $\bar{u}$ for the condenser $(S_\varepsilon, \Omega_R)$, when the length ε is 'small enough'.

According to the descending continuity property, it holds

$$\lim_{\varepsilon \to 0} C_p(S_\varepsilon, \Omega) = C_p(\{x_0\}, \Omega), \tag{5.4.3}$$

Hence in the case $p > N$, an idea could be to define $\bar{u}$ starting from the radial function $s_{p,N}$ minimizing the energy of condenser $(\{x_0\}, B(x_0, R))$. After Propositions 5.2.2 and 5.3.1, for all $x \in \bar{B}(x_0, R)$, denoting $r = |x - x_0|$, it holds

$$\begin{cases} s_{p,N}(x) = s_{p,N}(r) = 1 - (\frac{r}{R})^\beta, \\ |\nabla s_{p,N}(x)| = \frac{\beta}{R^\beta}\, r^{\beta-1}, x \neq x_0. \end{cases}$$

Moreover

$$\begin{aligned} C_p(\{x_0\}, B(x_0, R)) &= \int_{B(x_0,R)} |\nabla s_{p,N}(x)|^p \ dx \\ &= A^{N-1} \frac{\beta^p}{R^{\beta p}} \int_0^R r^{(\beta-1)p+N-1}\ dr = A^{N-1} \beta^{p-1} R^{N-p}, \end{aligned}$$

where exponent $(\beta - 1)p + N - 1 = \beta - 1 = -\frac{N-1}{p-1} \in (-1, 0)$.

Then consider function $\bar{u} : \overline{\Omega}_R \to \mathbb{R}$ defined by:

$$\begin{cases} \text{if } x \in \overline{S}_- \cap \{z < -\varepsilon/2\} & \text{then } \bar{u}(x) := s_{p,N}(\rho_-) \text{ with } \rho_- = |x - A| \ , \\ \text{if } x \in \overline{S}_+ \cap \{z > \varepsilon/2\} & \text{then } \bar{u}(x) := s_{p,N}(\rho_+) \text{ with } \rho_+ = |x - B| \ , \\ \text{if } x \in \overline{C} & \text{then } \bar{u}(x) := s_{p,N}(r) \text{ with } r = |y|. \end{cases}$$

as shown on Figure 5.4.

It is easy to check that $\bar{u}$ is continuous in $\overline{\Omega}_R$, and that $\bar{u} = 0$ on $\partial\Omega_R$ and $\bar{u} = 1$ on S_ε. As $\varepsilon > 0$ is 'small enough', one could expect function $\bar{u}$ to be an admissible function for condenser $(S_\varepsilon, \Omega_R)$. Moreover one could expect the energy of $\bar{u}$ to provide an approximation of the capacity $C_p(S_\varepsilon, \Omega_R)$, as

$$\int_{S_- \cup S_+} |\nabla \bar{u}(x)|^p \ dx = \int_{B(x_0,R)} |\nabla s_{p,N}(x)|^p \ dx = C_p(\{x_0\}, B(x_0, R)).$$

But the fact is function $\bar{u}$ is not even admissible for the condenser $(S_\varepsilon, \Omega_R)$. Calculating the energy of $\bar{u}$ in the cylindrical part C, it holds

$$\int_C |\nabla \bar{u}(x)|^p \ dx = \int_C |\nabla s_{p,N}(x)|^p \ dx = \varepsilon\ A^{N-2} \frac{\beta^p}{R^{\beta p}} \int_0^R r^{(\beta-1)p} r^{N-2}\ dr = +\infty,$$

as exponent $(\beta - 1)p + N - 2 \in (-2, -1)$.

Hence, despite the descending continuity property (5.4.3) in terms of energy, the solution minimizing the energy of the condenser, if it does exist, undergoes a sudden spatial reorganization when ε shifts from 0 to a positive value. What matters primarily here is not the length of the perturbation S_ε, but the discontinuity of its dimension from 0 to 1 at the very moment ε becomes positive. This example illustrates existing relationships between Hausdorff measures and capacities.

As a conclusion, when $p > N$, there is no hope to simply derive the asymptotic expansion of $C_p(S_\varepsilon, \Omega_R)$, even for $\varepsilon > 0$ small enough, from the knowledge we have about N-dimensional spherical condensers.

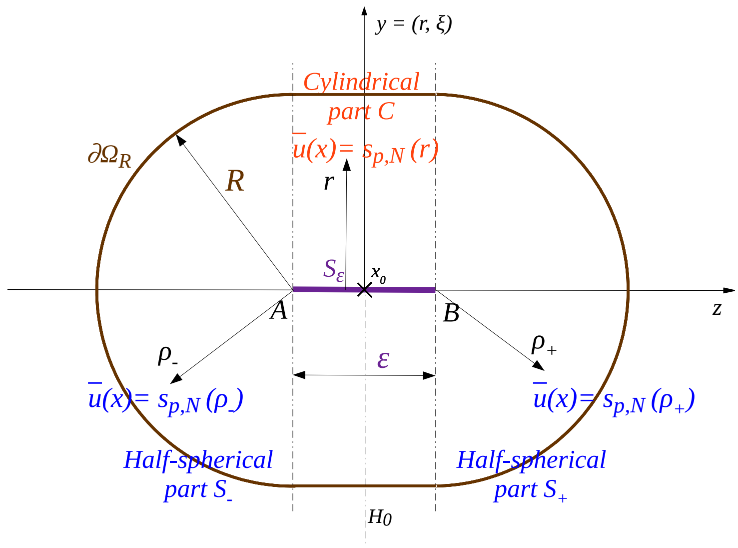

Figure 5.4: A poor candidate $\bar{u}$ for the condenser $(S_\varepsilon, \Omega_R)$.

5.4.4 A lower-bound to the p-capacity of a segment

Proposition 5.4.2. *With notations of subsection 5.4.1, the p-capacity of the equidistant condenser (K_η, Ω_R) admits the following lower-bound*

$$C_{p,N}\left(K_\eta, \Omega_R\right) \geq C_{p,N}(\eta, R) + \varepsilon\, C_{p,N-1}(\eta, R). \tag{5.4.4}$$

Proof. After subsection 5.2.1 and Proposition 5.2.1, we denote u the admissible function minimizing the energy for the condenser (K_η, Ω_R). As

$$C_{p,N}\left(K_\eta, \Omega_R\right) = \int_C |\nabla u|^p \, dx + \int_S |\nabla u|^p \, dx,$$

we estimate separately each integral.

1. In the cylindrical subset C, for all $a \in (-\varepsilon/2, \varepsilon/2)$, let w_a be the restriction of u to $H_a \cap \left(\overline{\Omega_R \setminus K_\eta}\right)$, that is

$$w_a(y) = u(a, y), \quad \forall y \in \mathbb{R}^{N-1}, \eta \leq |y| \leq R.$$

Due to the regularity of function u, w_a is defined pointwise, continuous in $H_a \cap (\overline{\Omega_R \setminus K_\eta})$ and w_a admits a classical gradient in $H_a \cap (\Omega_R \setminus K_\eta)$.
Since u is admissible for the condenser (K_η, Ω_R),
$|y| = \eta$ implies $w_a(y) = u(a, y) = 1$ and $|y| = R$ implies $w_a(y) = u(a, y) = 0$.
Moreover for all $y \in \mathbb{R}^{N-1}$, $\eta < |y| < R$ it holds:

$$|\nabla_y w_a(y)| = |\nabla_y u(a, y)| \leq \left[\, |\nabla_y u(a, y)|^2 + |\partial_z u(a, y)|^2 \,\right]^{1/2} = |\nabla u(a, y)|\,.$$

For a given $a \in (-\varepsilon/2, \varepsilon/2)$,

– if

$$\int_{H_a\cap(\Omega_R\setminus K_\eta)} |\nabla_y w_a(y)|^p \ dy \ < +\infty,$$

then w_a is admissible to the $(N-1)$-dimensional condenser $(\overline{B}_{N-1}(x_a,\eta), B_{N-1}(x_a,R))$. Thus:

$$C_{p,N-1}(\eta,R) \leq \int_{H_a\cap(\Omega_R\setminus K_\eta)} |\nabla_y \ w_a(y)|^p \ dy \leq \int_{H_a\cap(\Omega_R\setminus K_\eta)} |\nabla \ u(a,y)|^p \ dy. \tag{5.4.5}$$

– If

$$\int_{H_a\cap(\Omega_R\setminus K_\eta)} |\nabla_y w_a(y)|^p \ dy \ = +\infty,$$

inequality (5.4.5) obviously holds again.

Integrating inequality (5.4.5) for all $a \in (-\varepsilon/2, \varepsilon/2)$, one obtains

$$\varepsilon \ C_{p,N-1}(\eta,R) \leq \int_C |\nabla \ u(x)|^p \ dx. \tag{5.4.6}$$

2. Let v be the function defined in $\overline{B}(x_0,R)\setminus B(x_0,\eta)$ which inherits the values taken by u in the two half-spherical subsets $S_\pm$. More precisely, for all $x \in \mathbb{R}^N$, $\eta \leq |x-x_0| \leq R$, we define

$$\begin{cases} v(x) := u(B+x-x_0) & \text{if } z(x-x_0) \geq 0, \\ v(x) := u(A+x-x_0) & \text{if } z(x-x_0) < 0. \end{cases}$$

Since u is continuous in $\overline{\Omega_R\setminus K_\eta}$ and symmetric relatively to the hyperplane H_0, it follows that v is continuous in $\overline{B}(x_0,R)\setminus B(x_0,\eta)$. Similarly $u \in L^p\left(\Omega_R\setminus K_\eta\right)$ implies that $v \in L^p\left(B(x_0,R)\setminus \overline{B}(x_0,\eta)\right)$.

For all $x \in \left(B(x_0,R)\setminus \overline{B}(x_0,\eta)\right)\cap\{z\neq 0\}$ it holds

$$\begin{cases} \nabla v(x) = \nabla u(B+x-x_0) & \text{if } z(x-x_0) > 0, \\ \nabla v(x) = \nabla u(A+x-x_0) & \text{if } z(x-x_0) < 0. \end{cases}$$

Thus $\nabla u \in L^p(\Omega_R\setminus K_\eta)$ entails

$$\nabla v \in L^p\left(\left(B(x_0,R)\setminus \overline{B}(x_0,\eta)\right)\cap\{z>0\}\right)$$

and similarly

$$\nabla v \in L^p\left(\left(B(x_0,R)\setminus \overline{B}(x_0,\eta)\right)\cap\{z<0\}\right).$$

Moreover, since v is continuous in $\overline{B}(x_0,R)\setminus B(x_0,\eta)$ and thus has no jump across $\{z=0\}$, the results about distribution derivatives (e.g. [89]) entail that the distribution ∇v defined in the domain $\left(B(x_0,R)\setminus \overline{B}(x_0,\eta)\right)$ can be identified to the vector field $\{\nabla v\}$ defined in

$$\left(B(x_0,R)\setminus \overline{B}(x_0,\eta)\right)\cap\{z\neq 0\}.$$

Hence $\nabla v \in L^p\left(B(x_0,R)\setminus \overline{B}(x_0,\eta)\right)$.
Lastly, as u is admissible for the condenser (K_η,Ω_R), it holds $v(x)=1$ for all $x\in\mathbb{R}^N$, $|x-x_0|=\eta$ and $v(x)=0$ for all $x\in\mathbb{R}^N$, $|x-x_0|=R$.
Therefore v is an admissible function for the condenser $(\overline{B}(x_0,\eta),B(x_0,R))$. It follows that

$$C_{p,N}(\eta,R)\leq \int_{B(x_0,R)\setminus B(x_0,\eta)} |\nabla\, v(x)|^p\,dx = \int_S |\nabla\, u(x)|^p\,dx. \tag{5.4.7}$$

Summing inequalities (5.4.6) and (5.4.7) yields the claimed result

$$C_{p,N}\left(K_\eta,\Omega_R\right)\geq C_{p,N}(\eta,R)+\varepsilon\, C_{p,N-1}(\eta,R).$$

□

Thanks to equidistant condensers, we can now state the following lower-bound to the condenser p-capacity of a segment. Recall $C_{p,N}(\{x_0\},B_R)$ (resp. $C_{p,N-1}(\{x_0\},B_R)$ denotes the p-capacity of the point $\{x_0\}$ in the N-dimensional ball $B(x_0,R)$ (resp. the p-capacity of the point $\{x_0\}$ in the $(N-1)$-dimensional ball $B_{N-1}(x_0,R)$).

Theorem 5.4.3. *Let Ω be a bounded domain of $\mathbb{R}^N$ and $x_0\in\Omega$. Let*

$$R := \sup\{|y-x_0|\,;y\in\Omega\}\in(0,+\infty).$$

Let S_ε be a (closed) segment centered on the point x_0 and of length $\varepsilon>0$ such that $S_\varepsilon\subset\Omega$. Then the following lower-bound holds:

$$C_{p,N}(S_\varepsilon,\Omega)\ \geq\ C_{p,N}(\{x_0\},B_R)+\varepsilon\, C_{p,N-1}(\{x_0\},B_R). \tag{5.4.8}$$

Proof. For all $\lambda>0$ and all η such that $0<\eta<R$, inequality (5.4.4) of Proposition 5.4.2, applied to radii η and $R+\lambda$, reads:

$$C_{p,N}(\eta,R+\lambda)+\varepsilon\, C_{p,N-1}(\eta,R+\lambda)\leq C_{p,N}\left(K_\eta,\Omega_{R+\lambda}\right). \tag{5.4.9}$$

Three decreasing sequences of compacts are involved as follows:

$$\bigcap_{\eta>0}\overline{B}(x_0,\eta)=\{x_0\}\,,\ \bigcap_{\eta>0}\overline{B}_{N-1}(x_0,\eta)=\{x_0\}\ \text{ and }\ \bigcap_{\eta>0}K_\eta=S_\varepsilon.$$

The descending continuity property of Theorem 5.1.2 hence implies that:

$$\begin{cases}\lim\limits_{\eta\to0} C_{p,N}(\overline{B}(x_0,\eta),B(x_0,R+\lambda))=C_{p,N}(\{x_0\},B(x_0,R+\lambda))\\ \lim\limits_{\eta\to0} C_{p,N-1}(\overline{B}(x_0,\eta),B(x_0,R+\lambda))=C_{p,N-1}(\{x_0\},B(x_0,R+\lambda))\\ \lim\limits_{\eta\to0} C_{p,N}\left(K_\eta,\Omega_{R+\lambda}\right)=C_{p,N}(S_\varepsilon,\Omega_{R+\lambda}).\end{cases}$$

Therefore passing to the limit when $\eta\to0$ in inequality (5.4.9) yields

$$C_{p,N}(\{x_0\},B_{R+\lambda})+\varepsilon\, C_{p,N-1}(\{x_0\},B_{R+\lambda})\leq C_{p,N}\left(S_\varepsilon,\Omega_{R+\lambda}\right). \tag{5.4.10}$$

Moreover the inclusions $S_\varepsilon \subset \Omega \subset B(x_0, R+\lambda) \subset \Omega_{R+\lambda}$ hold. Hence the monotony property (ii) of Theorem 5.1.2 implies that

$$C_{p,N}(S_\varepsilon, \Omega_{R+\lambda}) \leq C_{p,N}(S_\varepsilon, B(x_0, R+\lambda)) \leq C_{p,N}(S_\varepsilon, \Omega). \tag{5.4.11}$$

Gathering inequalities (5.4.10) and (5.4.11) entails

$$C_{p,N}(\{x_0\}, B_{R+\lambda}) + \varepsilon\, C_{p,N-1}(\{x_0\}, B_{R+\lambda}) \leq C_{p,N}(S_\varepsilon, \Omega).$$

Lastly it follows from Theorem 5.3.1 that the mappings

$$R > 0 \mapsto C_{p,N}(\{x_0\}, B_R) \quad \text{and} \quad R \mapsto C_{p,N-1}(\{x_0\}, B_R)$$

are continuous. Hence letting λ tend towards 0 yields the claimed inequality. □

Remark 5.4.4. The lower-bound of Theorem 5.4.3 is worth interpreting. Recall from Proposition 5.3.1 that the capacity of point $\{x_0\}$ in a bounded ball of $\mathbb{R}^N$ is positive if and only if $p > N$. Accordingly three cases are to be considered:

– If $N-1 < p \leq N$, the point has a null N-dimensional condenser p-capacity but a positive $(N-1)$-dimensional condenser p-capacity. The inequality reads:

$$\varepsilon\, C_{p,N-1}(\{x_0\}, B_R) \leq C_{p,N}(S_\varepsilon, \Omega)$$

In particular, $C_{p,N}(S_\varepsilon, \Omega) > 0$.

– If $p > N$, both capacities $C_{p,N}(\{x_0\}, B_R)$ and $C_{p,N-1}(\{x_0\}, B_R)$ are positive. Then again $C_{p,N}(S_\varepsilon, \Omega) > 0$.

– If $p \leq N-1$, both capacities $C_{p,N}(\{x_0\}, B_R)$ and $C_{p,N-1}(\{x_0\}, B_R)$ are null.

Thus we can state a sufficient condition for the positivity of condenser p-capacities of segments.

Corollary 5.4.5. *Let S_ε be a segment of length $\varepsilon > 0$ included in a bounded domain $\Omega \subset \mathbb{R}^N$. If $p > N-1$ then $C_{p,N}(S_\varepsilon, \Omega) > 0$.*

5.4.5 Positivity of a condenser p-capacity of a segment in a bounded domain

Proposition 5.4.6. *Let $S_\varepsilon \subset \Omega$ be a segment of length $\varepsilon > 0$ centered on a point x_0. If $p \leq N-1$, then the condenser p-capacity of the segment S_ε in the domain Ω is null, that is $C_{p,N}(S_\varepsilon, \Omega) = 0$.*

The proof is available in section 5.5.2 on page 136. According to Corollary 5.4.5 and to Proposition 5.4.6, we can state the following positivity rule for condenser p-capacities of segments.

Proposition 5.4.7. *The condenser p-capacity of a segment S_ε of length $\varepsilon > 0$ included in a bounded domain $\Omega \subset \mathbb{R}^N$ is positive if and only if $p > N-1$.*

Remark 5.4.8. It appeared according to Proposition 5.4.2 that the positivity of the condenser p-capacity of a segment in a N-dimensional bounded domain follows from the positivity of the condenser p-capacity of a point in a $(N-1)$-dimensional bounded domain.

Using equidistant condensers derived from a *plane rectangle*, we may think of a similar proof to show that the positivity of the p-capacity of a plane rectangle in a N-dimensional bounded domain follows from the positivity of the p-capacity of a segment in a $(N-1)$-dimensional bounded domain, which happens when $p > (N-1)-1$. Such reasoning can be extended by induction to prove that the condenser p-capacity of a k-dimensional closed box in a N-dimensional bounded domain is positive as soon as $p > N - k$.

The cases of nullity for condenser capacity of a k-dimensional closed box seem to be more intricate to establish by means of equidistant condensers as the relationship between the capacity of a segment in a N-dimensional domain and the capacity of a point in a $(N-1)$-dimensional domain is not straightforward in the proof of Proposition 5.4.6.

5.4.6 Elliptical condensers

Let again a (closed) segment $S_\varepsilon \subset \mathbb{R}^N$ $(N \geq 2)$, of length $\varepsilon > 0$ and centered on a point x_0. Let z be an axis passing through the point x_0 and parallel to the segment S_ε.

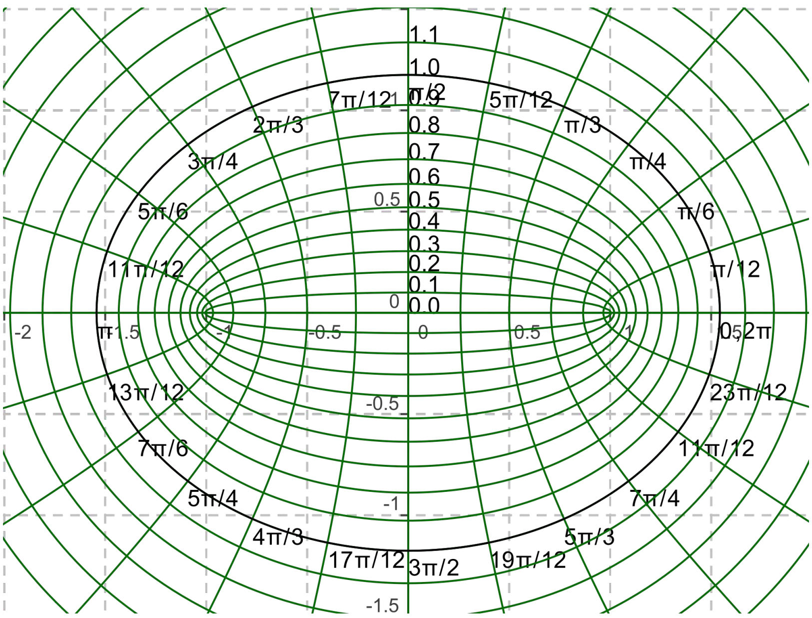

Figure 5.5: Elliptic coordinates (source Wikipedia [93]).

We first consider cylindrical coordinates $(z, y) = (z, r, \xi)$, with $z \in \mathbb{R}$, $y = r\xi \in \mathbb{R}^{N-1}$, $r \geq 0$ and $\xi \in S^{N-2}$. Then we move forward to the elliptic coordinates (μ, ν, ξ)

[92, 93] implicitly defined as follows for $\mu \in [0, +\infty)$, $\nu \in [0, \pi]$ and $\xi \in S^{N-2}$,

$$\begin{cases} z(\mu,\nu) & := \frac{\varepsilon}{2} \cosh \mu \cos \nu, \\ r(\mu,\nu) & := \frac{\varepsilon}{2} \sinh \mu \sin \nu, \\ \xi & := \xi. \end{cases} \tag{5.4.12}$$

In particular $S_\varepsilon = \{\mu = 0, \nu \in [0, \pi]\}$. Hence the segment S_ε is considered here as the limit of an ellipsoid which eccentricity tends toward 1 when $\mu \to 0$.

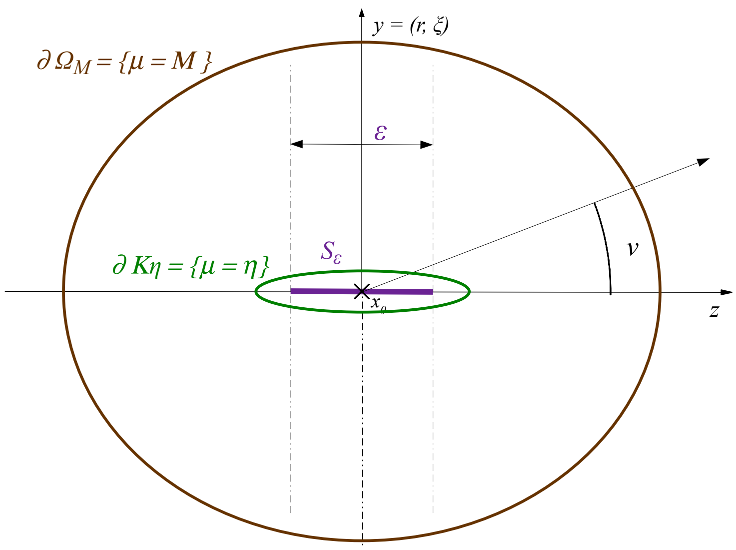

Figure 5.6: An elliptical condenser (K_η, Ω_M).

Looking at Figure 5.6 we set

Definition 5.4.9. Let $0 < \eta < M$. Let the bounded domain

$$\Omega_M := \left\{ x = (\mu, \nu, \xi) \in \mathbb{R}^N \ ; \ 0 \le \mu < M, \nu \in [0, \pi], \xi \in S^{N-2} \right\}$$

and the compact set

$$K_\eta := \left\{ x = (\mu, \nu, \xi) \in \mathbb{R}^N \ ; \ 0 \le \mu \le \eta, \nu \in [0, \pi], \xi \in S^{N-2} \right\}.$$

We say that (K_η, Ω_M) is an *elliptical condenser derived from the segment* S_ε.

Obviously the inclusions $S_\varepsilon \subset K_\eta \subset \Omega_M$ hold for all $0 < \eta < M$. Moreover it holds $\bigcap_{\eta > 0} K_\eta = S_\varepsilon$. In comparison with equidistant condensers though, letting $\eta \to 0$ will not be sufficient to approximate asymptotically, when $\varepsilon \to 0$, the condenser made of the segment S_ε within a given bounded domain Ω. Due to (5.4.12) indeed , for a given $M > 0$, $\Omega_M \to \{x_0\}$ when $\varepsilon \to 0$. So that we shall have to choose some appropriate $M(\varepsilon) \to +\infty$ to approximate a given domain Ω by $\Omega_{M(\varepsilon)}$ when $\varepsilon \to 0$.

Lemma 5.4.10. *Let $R > \varepsilon/2 > 0$ and let*

$$\begin{cases} M' := \log\left(2R/\varepsilon + \sqrt{1 + 4R^2/\varepsilon^2}\right) \\ M'' := \log\left(2R/\varepsilon + \sqrt{-1 + 4R^2/\varepsilon^2}\right). \end{cases} \tag{5.4.13}$$

Let K a compact of $\mathbb{R}^N$ such that $K \subset \Omega_{M''}$. Then it holds

$$C_p(K, \Omega_{M'}) \leq C_p(K, B_R) \leq C_p(K, \Omega_{M''}). \tag{5.4.14}$$

In particular, for all η such that $0 < \eta < M''$, it holds

$$C_p(K_\eta, \Omega_{M'}) \leq C_p(K_\eta, B_R) \leq C_p(K_\eta, \Omega_{M''}). \tag{5.4.15}$$

and

$$C_p(S_\varepsilon, \Omega_{M'}) \leq C_p(S_\varepsilon, B_R) \leq C_p(S_\varepsilon, \Omega_{M''}) \tag{5.4.16}$$

Proof. It follows from (5.4.12) that

$$B_{\frac{\varepsilon}{2}\sinh M} \subset \Omega_M \subset B_{\frac{\varepsilon}{2}\cosh M}, \quad \forall M > 0. \tag{5.4.17}$$

Moreover the definition (5.4.13) of M' and M'' can be solved into

$$R = \frac{\varepsilon}{2}\, \sinh M' = \frac{\varepsilon}{2}\, \cosh M''. \tag{5.4.18}$$

Hence plugging $M = M'$ (resp. $M = M''$) into (5.4.17), one obtains

$$\Omega_{M''} \subset B_R \subset \Omega_{M'}.$$

Thus the monotony property (ii) of Theorem 5.1.2 implies

$$C_p(K, \Omega_{M'}) \leq C_p(K, B_R) \leq C_p(K, \Omega_{M''}) \tag{5.4.19}$$

which the claimed inequality (5.4.14).

In particular, choosing $K = K_\eta$ (resp. $K = S_\varepsilon$), it follows (5.4.15) (resp.(5.4.16)). □

5.4.7 The condenser 2-capacity of a segment

In the harmonic case $p = 2$, according to Theorem 5.4.7, the condenser capacity of a segment is positive in a bounded domain of $\mathbb{R}^2$ while it is null in higher dimensions $N \geq 3$.

Proposition 5.4.11. *Let $0 < \varepsilon/2 < R$. Let S_ε a (closed) segment centered on a point x_0 and of length ε and let $B_R = B(x_0, R)$ be both subsets of $\mathbb{R}^2$. Then the following inequalities hold:*

$$\frac{2\pi}{\log\left(2R/\varepsilon + \sqrt{1 + 4R^2/\varepsilon^2}\right)} \leq C_2(S_\varepsilon, B_R) \leq \frac{2\pi}{\log\left(2R/\varepsilon + \sqrt{-1 + 4R^2/\varepsilon^2}\right)}. \tag{5.4.20}$$

Proof. We calculate $C_p(K_\eta, \Omega_M)$ applying Proposition 5.2.1. Due to the symmetry of revolution relatively to the z-axis, the searched solution does not depend upon coordinate $\xi \in S^{N-2}$. Thus in elliptic coordinates, the Laplacian operator applied to u reads:

$$\Delta u(\mu, \nu) = \frac{4}{\varepsilon^2} \frac{\partial_{\mu\mu} u + \partial_{\nu\nu} u}{\sinh^2 \mu + \sin^2 \nu}.$$

Hence Problem (5.2.1) becomes

$$\begin{cases} \partial_{\mu\mu} u + \partial_{\nu\nu} u = 0 & \text{in } \Omega_M \setminus K_\eta, \\ u(\eta, \nu) = 1 & \forall \nu \in [0, \pi], \\ u(M, \nu) = 0 & \forall \nu \in [0, \pi]. \end{cases} \tag{5.4.21}$$

The separation of variables yields

$$u(\mu, \nu) = \frac{M - \mu}{M - \eta}, \quad \forall \mu \in [\eta, M], \ \forall \nu \in [0, \pi].$$

Then a simple calculation provides

$$|\nabla u|^2 = \frac{4}{\varepsilon^2 \left(\sinh^2 \mu + \sin^2 \nu\right)} \frac{1}{(M - \eta)^2}.$$

Since

$$|\det D(z, r, \xi)/D(\mu, \nu, \xi)| = (\varepsilon/2)^2 \left(\sinh^2 \mu + \sin^2 \nu\right),$$

the change of variables yields

$$C_2(K_\eta, \Omega_M) = \int_{\Omega_M \setminus K_\eta} |\nabla u|^2 = \frac{2\pi}{M - \eta}.$$

When $\eta \to 0$, according to the descending continuity stated in Theorem 5.1.2 it follows that

$$C_2(S_\varepsilon, \Omega_M) = \frac{2\pi}{M}. \tag{5.4.22}$$

Applying equality (5.4.22) for $M = M'$ (resp. $M = M''$) and inequality (5.4.16) of Lemma 5.4.10 one obtains

$$\frac{2\pi}{\log\left(2R/\varepsilon + \sqrt{1 + 4R^2/\varepsilon^2}\right)} \leq C_2(S_\varepsilon, B_R) \leq \frac{2\pi}{\log\left(2R/\varepsilon + \sqrt{-1 + 4R^2/\varepsilon^2}\right)} \tag{5.4.23}$$

which are the claimed inequalities (5.4.20). □

Corollary 5.4.12. *Let Ω a bounded domain $\subset \mathbb{R}^2$ and x_0 a point of Ω. Let $S_\varepsilon \subset \mathbb{R}^2$ be a segment centered on a point x_0 and of length $\varepsilon > 0$. Then for ε small enough it holds*

$$C_2(S_\varepsilon, \Omega) = \frac{-2\pi}{\log \varepsilon} + o\left(\frac{-1}{\log \varepsilon}\right). \tag{5.4.24}$$

Proof. Let $R_2 > R_1 > 0$ such that $B(x_0, R_1) \subset \Omega \subset B(x_0, R_2)$. Let $0 < \varepsilon < R_1$. Thus $S_\varepsilon \subset B(x_0, R_1)$. According to monotony property (ii) of Theorem 5.1.2, it holds

$$C_2(S_\varepsilon, B(x_0, R_2)) \leq C_2(S_\varepsilon, \Omega) \leq C_2(S_\varepsilon, B(x_0, R_1)).$$

Applying inequalities (5.4.20) of Proposition 5.4.11 for $R = R_1$ (resp. $R = R_2$), one obtains

$$C_2(S_\varepsilon, B(x_0, R_1)) \leq \frac{2\pi}{\log\left(2R_1/\varepsilon + \sqrt{-1 + 4R_1^2/\varepsilon^2}\right)}$$

and

$$\frac{2\pi}{\log\left(2R_2/\varepsilon + \sqrt{1 + 4R_2^2/\varepsilon^2}\right)} \leq C_2(S_\varepsilon, B(x_0, R_2)).$$

Hence

$$\frac{2\pi}{\log\left(2R_2/\varepsilon + \sqrt{1 + 4R_2^2/\varepsilon^2}\right)} \leq C_2(S_\varepsilon, \Omega) \leq \frac{2\pi}{\log\left(2R_1/\varepsilon + \sqrt{-1 + 4R_1^2/\varepsilon^2}\right)}.$$

The claimed asymptotic expansion (5.4.24) follows. □

Remark 5.4.13.
- Asymptotic expansion (5.4.24) is a topological asymptotic expansion in the sense of (1.1.1), here obtained with Dirichlet boundary condition. For the study of the perturbation of the Laplace equation in 2D by a Neumann homogeneous boundary condition on a segment, see [9].
- According to [47], it is already known that, in the case of the Laplace equation with Dirichlet boundary condition in 2D, the first order of the topological asymptotic expansion does not depend on the shape of the obstacle, in the case of obstacles with *non-empty interiors*.

  After Proposition 5.2.2 in the case $p = N = 2$ and for spherical condensers, it holds

  $$C_2(B_{\varepsilon/2}, B_R) = \frac{2\pi}{\log(2R/\varepsilon)} = \frac{-2\pi}{\log\varepsilon} + o\left(\frac{-1}{\log\varepsilon}\right).$$

  It is noticeable that the first term $-2\pi/\log\varepsilon$ of the asymptotic expansion (5.4.24) of $C_2(S_\varepsilon, B_R)$ is the same as the first term of the expansion of $C_2(B_{\varepsilon/2}, B_R)$. Thus according to monotony property (i) of Theorem 5.1.2, it holds

  $$C_2(K_\varepsilon, B_R) = \frac{-2\pi}{\log\varepsilon} + o\left(\frac{-1}{\log\varepsilon}\right), \text{ for all compact } K_\varepsilon,\ S_\varepsilon \subset K_\varepsilon \subset B_{\varepsilon/2}. \quad (5.4.25)$$

  Hence the topological gradient does not depend on the shape of the obstacle K_ε, even when K_ε is the segment S_ε. It only depends on its size. Therefore the topological gradient of the 2-capacity in 2D is not a appropriate tool for sorting out curves and obstacles with non empty interior.

  One could try to overcome this drawback, either by considering different values of p or by trying to sort out shapes according to the second order of the asymptotic expansion, that is considering a *topological hessian*.

  Though we do not know yet the asymptotic expansion of $C_p(S_\varepsilon, B_R)$ in higher dimensions $N \geq 3$, the same difficulty might also arise for $p = N \geq 3$ according to Remark 5.2.4 on page 118.

Remark 5.4.14. In the harmonic case $p = 2$, equation (5.2.1) can be explicitly solved as its enjoys separable variables in elliptic coordinates. However there are some clues that, even when $p \neq 2$, elliptical condensers may be applied for further estimations of the p-capacity of a segment S_ε in a given domain Ω. As usual, S_ε has first to be approximated by an ellipsoid K_η, with η 'small enough'.

Elliptic coordinates seem appropriate for segments in the sense that angular coordinate ν so to speak makes the dimension in which operates the p-Laplace equation, continuously change from N for $\nu = 0$ to $(N-1)$ for $\nu = \pi/2$ and then back to N for $\nu = \pi$.

Furthermore the geometry of the problem is simplified with elliptical condensers as solutions and integrals are to be calculated on the rectangle $\mathcal{R} := [\eta, M] \times [0, \pi]$.

After a change of variables, one can rewrite minimization problem (5.2.3) in elliptic coordinates $(\mu, \nu) \in \mathcal{R}$. Let the weight E given by

$$E(\mu,\nu) := \frac{(\sinh\mu \sin\nu)^{N-2}}{\left(\sinh^2\mu + \sin^2\nu\right)^{\frac{p-2}{2}}},\ \forall \mu > 0,\ \forall \nu \in [0,\pi],$$

and the functional

$$J(v) := A^{N-2}\left(\frac{\varepsilon}{2}\right)^{N-p} \int_{\mathcal{R}} E(\mu,\nu)\,|\nabla v(\mu,\nu)|^p \,\mathrm{d}\mu\,\mathrm{d}\nu.$$

Then the capacity $C_p(K_\eta, \Omega_M)$ may be obtained by minimizing functional J in an appropriate set of functions $v : \mathcal{R} \to \mathbb{R}$, such that $E^{\frac{1}{p}}\ \nabla v \in L^p(\mathcal{R})$ and such that $v(\eta,\nu) = 1$ and $v(M,\nu) = 0$, for all $\nu \in [0,\pi]$.

5.5 Proofs

5.5.1 Proof of Proposition 5.2.2

We prove Proposition 5.2.2 by solving problem (5.2.1) for the spherical condenser and then applying Proposition 5.2.1. Hence consider the following problem:

$$\begin{cases} -\Delta_p\ s_{p,N} = 0 & \text{in } B_R \setminus \bar{B}_\varepsilon\ , \\ s_{p,N} = 1 & \text{on } \partial B_\varepsilon\ , \\ s_{p,N} = 0 & \text{on } \partial B_R\ . \end{cases} \tag{5.5.1}$$

As recalled in subsection 5.2.1 on page 113, there exists a unique solution $s_{p,N} \in W^{1,p}(B_R \setminus \bar{B}_\varepsilon)$ to problem (5.5.1). We look for a solution $s_{p,N}$ with radial symmetry, that is:

$$s_{p,N}(x) = f(r), \quad \forall x \in B_R \setminus \bar{B}_\varepsilon, r = |x - x_0|\,.$$

Assuming that function f is regular enough, a calculation in spherical coordinates yields:

$$\Delta_p\ s_{p,N} = \pm(f'(r))^{(p-1)}[(p-1)f''(r) + \frac{N-1}{r}f'(r)]. \tag{5.5.2}$$

We assume that there is no point in $\bar{B}_R \setminus B_\varepsilon$ such that $f'(r) = 0$.

1. In the case $p = N$, it follows that $f'(r) = C/r$. Then applying the Dirichlet conditions of problem (5.5.1) yields:

$$s_{p,N}(x) = f(r) = \frac{\log(R/r)}{\log(R/\varepsilon)}, \quad \forall x \in B_R \setminus \bar{B}_\varepsilon \tag{5.5.3}$$

and

$$|\nabla s_{p,N}(x)| = |f'(r)| = \frac{1}{\log(R/\varepsilon)} \frac{1}{r}, \quad \forall x \in B_R \setminus \bar{B}_\varepsilon. \tag{5.5.4}$$

Conversely it is easy to check that there is no point in $\bar{B}_R \setminus B_\varepsilon$ such that $|\nabla s_{p,N}(x)| = 0$ as defined in (5.5.4) and that the function $s_{p,N}$ stated by (5.5.3) is the unique solution to problem (5.5.1).

Applying Proposition 5.2.1, the capacity of the spherical condenser $(\overline{B}_\varepsilon, B_R)$ reads:

$$C_p(\varepsilon, R) = \frac{A^{N-1}}{(\log(R/\varepsilon))^p} \int_\varepsilon^R \frac{r^{N-1}}{r^p}\, dr = A^{N-1} \left[\log(R/\varepsilon)\right]^{1-p}.$$

The claimed asymptotic expansion follows when $\varepsilon \to 0$.

2. In the case $p \neq N$, it follows from (5.5.2) that $f'(r) = Cr^{\beta-1}$. Applying the Dirichlet conditions of problem (5.5.1) yields

$$s_{p,N}(x) = f(r) = \frac{R^\beta - r^\beta}{R^\beta - \varepsilon^\beta}, \quad \forall x \in B_R \setminus \bar{B}_\varepsilon, \tag{5.5.5}$$

and

$$|\nabla s_{p,N}(x)| = |f'(r)| = \left| \frac{\beta}{R^\beta - \varepsilon^\beta} \right| \; r^{\beta-1}, \quad \forall x \in B_R \setminus \bar{B}_\varepsilon. \tag{5.5.6}$$

Conversely it is easy to check that there is no point in $\bar{B}_R \setminus B_\varepsilon$ such that $|\nabla s_{p,N}(x)| = 0$ as defined in equation (5.5.6) and that the function $s_{p,N}$ defined by equation (5.5.5) is the unique solution to problem (5.5.1).

Applying Proposition 5.2.1, the capacity of the spherical condenser $(\overline{B}_\varepsilon, B_R)$ reads:

$$C_p(\overline{B}_\varepsilon, B_R) = A^{N-1} \left| \frac{\beta}{R^\beta - \varepsilon^\beta} \right|^p \int_\varepsilon^R r^{N-1+p(\beta-1)}\, dr = A^{N-1} |\beta|^{p-1} \left| R^\beta - \varepsilon^\beta \right|^{1-p}. \tag{5.5.7}$$

The claimed asymptotic expansions follow, when $\varepsilon \to 0$, dealing separately with the two cases $p > N$ and $p < N$.

Note that (5.5.5), (5.5.6) and (5.5.7) also hold in the case $N = 1 < p$, with the convention $A^0 = 2$.

5.5.2 Proof of Proposition 5.4.6

Let $\Omega^c := \mathbb{R}^N \backslash \Omega$. Since Ω is bounded there exists $M > 0$ such that $\Omega \subset B(x_0, M/2)$. Then S_ε and $\Omega^c \cap \overline{B}(x_0, M)$ are compact sets such that

$$S_\varepsilon \cap \left(\Omega^c \cap \overline{B}(x_0, M)\right) = \emptyset.$$

Therefore due to the continuity of the distance, there exist

$$x' \in S_\varepsilon \quad \text{and} \quad x'' \in \Omega^c \cap \overline{B}(x_0, M)$$

such that:

$$|x' - x''| = \min\left\{|x_1 - x_2|\,; x_1 \in S_\varepsilon \text{ and } x_2 \in \Omega^c \cap \overline{B}(x_0, M)\right\} > 0.$$

Let $R := |x' - x''|/2$ and $\Omega_R = \left\{x \in \mathbb{R}^N; d(S_\varepsilon, x) < R\right\}$. It holds $S_\varepsilon \subset \Omega_R \subset \Omega$. Hence according to monotony property (ii) stated in Theorem 5.1.2, it follows that

$$C_{p,N}(S_\varepsilon, \Omega) \leq C_{p,N}(S_\varepsilon, \Omega_R).$$

Therefore it suffices to prove that $C_{p,N}(S_\varepsilon, \Omega_R) = 0$. According to the descending continuity property of Theorem 5.1.2, it holds

$$C_{p,N}(S_\varepsilon, \Omega_R) = \lim_{\eta \to 0} C_p(K_\eta, \Omega_R).$$

Hence it suffices to prove that

$$\lim_{\eta \to 0} C_p(K_\eta, \Omega_R) = 0. \tag{5.5.8}$$

1. We first prove (5.5.8) in the case $p < N - 1$.
   Recall $s_{p,N}$ denotes the admissible function minimizing energy for the N-dimensional spherical condenser (B_η, B_R), that is

   $$C_{p,N}(B_\eta, B_R) = \int_{B_R \backslash B_\eta} |\nabla s_{p,N}|^p \, dx.$$

   Let the function $v : \overline{\Omega_R \setminus K_\eta} \to \mathbb{R}$ defined by:

   $$\begin{cases} \text{if } x \in \overline{S}_- \cap \{z < -\varepsilon/2\} & \text{then } v(x) := s_{p,N}(\rho_-) \text{ with } \rho_- = |x - A|\ , \\ \text{if } x \in \overline{S}_+ \cap \{z > \varepsilon/2\} & \text{then } v(x) := s_{p,N}(\rho_+) \text{ with } \rho_+ = |x - B|\ , \\ \text{if } x \in \overline{C} & \text{then } v(x) := s_{p,N}(r) \text{ with } r = |y|. \end{cases}$$

   as shown on Figure 5.7.
   It is easy to check that v is continuous in $\overline{\Omega_R \setminus K_\eta}$, that $v \in W^{1,p}(\Omega_R \setminus K_\eta)$ and that $v = 0$ on $\partial\Omega_R$ and $v = 1$ on ∂K_η. Thus v is admissible for the condenser (K_η, Ω_R). Hence

   $$C_{p,N}\left(K_\eta, \Omega_R\right) \leq \int_{C \cup S} |\nabla v|^p \, dx.$$

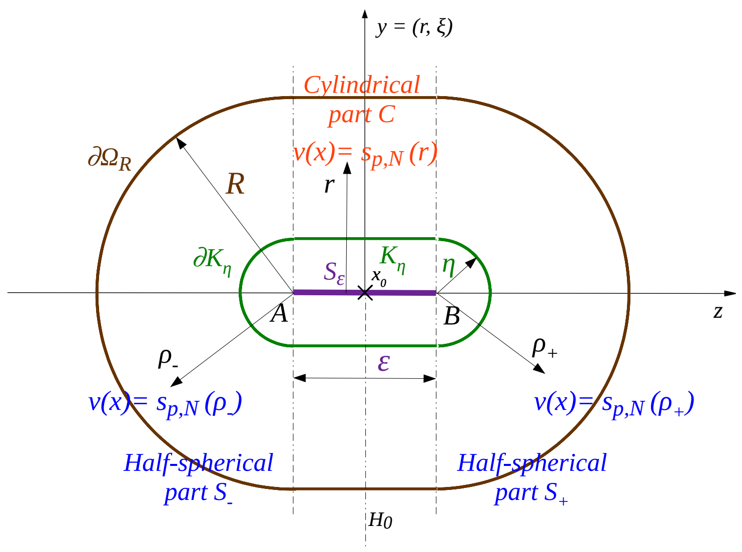

Figure 5.7: The admissible function v for the condenser (K_η, Ω_R).

Thus to prove (5.5.8) it suffices to prove that

$$\lim_{\eta \to 0} \int_{C \cup S} |\nabla v|^p \, dx = 0. \tag{5.5.9}$$

By definition of v it holds

$$\int_S |\nabla v|^p \, dx = C_{p,N}(B_\eta, B_R).$$

Since $p < N$, it follows from the descending continuity of Theorem 5.1.2 and from Proposition 5.3.1 that

$$\lim_{\eta \to 0} \int_S |\nabla v|^p \, dx = \lim_{\eta \to 0} C_{p,N}(B_\eta, B_R) = C_{p,N}(\{x_0\}, B_R) = 0.$$

Furthermore an integration in cylindrical coordinates in C yields:

$$\int_C |\nabla v|^p \, dx = \varepsilon \, A^{N-2} \int_\eta^R |\partial_r s_{p,N}(r)|^p r^{N-2} \, dr$$

As $p < N-1$ after Proposition 5.2.2 it holds

$$|\partial_r s_{p,N}(r)| = \left[-\beta/(\eta^\beta - R^\beta) \right] \, r^{\beta-1}.$$

Hence

$$\int_\eta^R |\partial_r s_{p,N}(r)|^p r^{N-2} \, dr \; = \left[\frac{-\beta}{\eta^\beta - R^\beta} \right]^p \, \frac{\eta^{\beta-1} - R^{\beta-1}}{1-\beta}.$$

Since $\beta < 0$, when η tends towards 0, the integral is equivalent to

$$\frac{(-\beta)^p}{1-\beta} \, \eta^{\beta - 1 - p\beta}$$

with $\beta - 1 - p\beta = N - p - 1 > 0$. It follows that

$$\lim_{\eta\to 0} \int_\eta^R |\partial_r s_{p,N}(r)|^p r^{N-2}\, dr = 0$$

and that $\lim\limits_{\eta\to 0} \int_C |\nabla v|^p \; dx = 0$.
This completes the proof of (5.5.9) and thus the proof of (5.5.8) in the case $p < N-1$.

2. We now prove (5.5.8) in the case $p = N-1$.
Recall $s_{p,N-1}$ denotes the admissible function minimizing energy for the $(N-1)$-dimensional spherical condenser $(B_{N-1}(\eta), B_{N-1}(R))$, that is

$$C_{p,N-1}(B_{N-1}(\eta), B_{N-1}(R)) = \int_{B_{N-1}(R)\setminus B_{N-1}(\eta)} |\nabla s_{p,N-1}|^p \; dy,$$

where dy denotes the Lebesgue measure in $\mathbb{R}^{N-1}$.
Let the function $w : \Omega_R \setminus K_\eta \to \mathbb{R}$ defined by:

$$\begin{cases} \text{if } x \in \overline{C} & \text{then } w(x) := s_{p,N-1}(r) \text{ with } r = |y|, \\ \text{if } x \in \overline{S}_- \cap \{z < -\varepsilon/2\} & \text{then } w(x) := s_{p,N-1}(\rho_-) \text{ with } \rho_- = |x - A|\,, \\ \text{if } x \in \overline{S}_+ \cap \{z > \varepsilon/2\} & \text{then } w(x) := s_{p,N-1}(\rho_+) \text{ with } \rho_+ = |x - B|\,. \end{cases}$$

as shown on Figure 5.8.
As for function v, it is easy to check that w is an admissible function for the N-dimensional condenser (K_η, Ω_R). Hence

$$C_{p,N}\left(K_\eta, \Omega_R\right) \leq \int_{C\cup S} |\nabla w|^p \; dx.$$

Thus to prove (5.5.8), it suffices to prove that

$$\lim_{\eta\to 0} \int_{C\cup S} |\nabla w|^p \; dx = 0. \tag{5.5.10}$$

By definition of w it holds

$$\int_C |\nabla w|^p \; dx = \varepsilon\; C_{p,N-1}(B_{N-1}(\eta), B_{N-1}(R)).$$

Since $p = N-1$, it follows from the descending continuity of Theorem 5.1.2 and from Theorem 5.3.1 that

$$\begin{aligned} \lim_{\eta\to 0} \int_C |\nabla w|^p \; dx &= \varepsilon \lim_{\eta\to 0} C_{p,N-1}(B_{N-1}(\eta), B_{N-1}(R)) \\ &= \varepsilon\; C_{p,N-1}(\{x_0\}, B_{N-1}(R)) = 0. \end{aligned}$$

Furthermore an integration in spherical coordinates in S yields:

$$\int_S |\nabla w|^p \; dx = A^{N-1} \int_\eta^R |\partial_\rho s_{p,N-1}(\rho)|^p \rho^{N-1}\, d\rho.$$

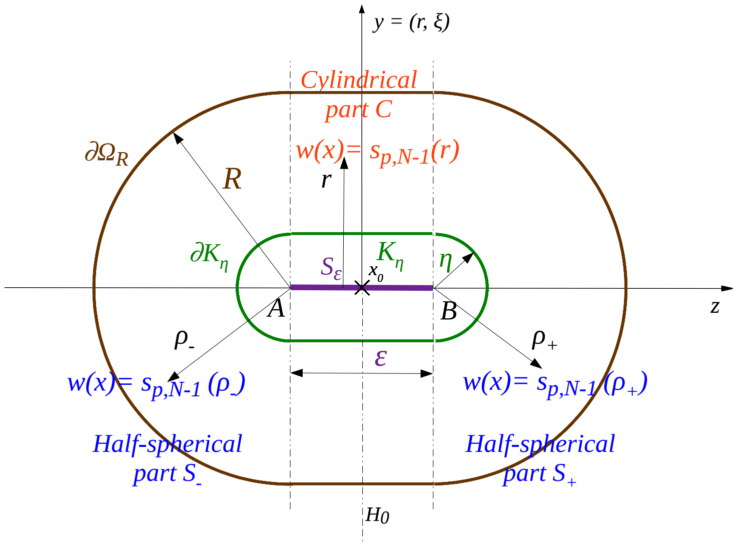

Figure 5.8: The admissible function w for the condenser (K_η, Ω_R).

As $p = N-1$, the gradient reads

$$|\partial_\rho s_{p,N-1}(\rho)| = \frac{1}{\log(R/\eta)}\,\frac{1}{\rho}.$$

Hence

$$\int_\eta^R |\partial_\rho s_{p,N-1}(\rho)|^p \rho^{N-1}\,d\rho = |\log(R/\eta)|^{-p}\ (R-\eta).$$

Therefore

$$\lim_{\eta\to 0}\int_\eta^R |\partial_\rho s_{p,N-1}(\rho)|^p \rho^{N-1}\,d\rho = 0$$

and thus

$$\lim_{\eta\to 0}\int_S |\nabla w|^p\ dx\ = 0.$$

This completes the proof of (5.5.10) and thus the proof (5.5.8) in the case $p = N-1$.

5.6 Conclusion

In Part II, we have considered compact sets with positive condenser p-capacities. We have provided estimates and asymptotic expansions of such capacities, focusing on capacities of points and of segments.

Our main contributions deal with condenser p-capacities of segments. Nevertheless it is fruitful to first study in detail condenser capacities which obstacle either has a non empty interior or is a point.

As preliminary results, we show that one can calculate a condenser p-capacity by solving a p-Laplace equation with Dirichlet boundary condition, when the boundaries of the condenser are smooth. We provide asymptotic bounds to condenser p-capacities when the obstacle has a non-empty interior. With respect to condenser p-capacities of points, we provide estimates, asymptotic approximations and positivity cases. We study the speed of the descending continuity property, i.e. the ability to approximate the capacity of a point by that of a ball with a 'small enough' radius. We show that the speed of convergence is slow.

We then move on to condenser p-capacities of segments. Introducing *equidistant condensers*, we illustrate in two ways the strong anisotropy caused by a segment acting as an obstacle in the p-Laplace equation. First we show that the Pólya-Szegö rearrangement inequality for Dirichlet type integrals fails to provide a valuable lower-bound to the capacity of a segment. Secondly, in the case $p > N$, we show that, however small the length of a segment may be, one cannot derive an admissible function for the segment by simply extending the function minimizing the energy for a point.

Our main contribution is to provide a lower bound to the N-dimensional condenser p-capacity of the segment which brings into light its relationship with the N-dimensional capacity of a point and more significantly with the $(N-1)$-dimensional capacity of a point. Our lower bound allows to establish the positivity rule for condenser p-capacity of a segment. This new method can be extended to study the p-capacity of a plane rectangle and by induction for closed boxes in higher dimensions.

Introducing *elliptical condensers*, we provide an estimate for the condenser 2-capacity of a segment in the plane. The asymptotic expansion follows when the length of the segment goes down to zero. Comparing with results obtained for spherical condensers, it turns out that the topological gradient of the 2-capacity is not an appropriate tool to separate curves and obstacles with non-empty interior in 2D. One way out could be to consider different values of parameter p. Another way might be to sort out shapes of obstacles according to the second order of the topological asymptotic expansion, that is considering the *topological hessian*.

When $p \neq 2$, elliptical condensers may prove useful to further obtain estimates of condenser p-capacities of segments.

In the wake of Hausdorff's definition of non-integer dimensions, the p-capacity of a segment offers a practical study case of a differential operator operating so to speak in between two dimensions, related to two orthogonal directions.

On the basis of this chapter, research about estimates and asymptotic expansions of condenser p-capacities of obstacles with empty interior, may now develop in several directions:

– the study of the convergence speed of descending continuity, in the case the limit compact is a segment or more generally any compact with empty interior;

- the search for more powerful tools resorting to the field of measure transportation, which could be able to grasp the anisotropy caused by the segment;
- the development of methods based on elliptical condensers to obtain further estimations of capacities of segments;
- the ability of the asymptotic expansion of a p-capacity, at first or second order, to sort out curves and obstacles with non-empty interior in 2D and in $3D$;
- the study of similar questions for other obstacles in higher dimensions, such as surfaces in $\mathbb{R}^4$.

Troisième partie

Synthèse en langue française

Cette partie présente de façon synthétique en langue française les travaux réalisés et les résultats obtenus au cours de ce travail de thèse.

Ce manuscrit étudie et introduit plusieurs résultats nouveaux concernant :

1. les développements asymptotiques topologiques pour des équations elliptiques quasilinéaires (non linéaires), perturbées dans des sous-domaines non vides ;
2. les estimations et les développements asymptotiques de p-capacités de condensateurs, en particulier pour des obstacles d'intérieur vide, notamment dans le cas anisotrope du segment.

Ces travaux ont en commun l'obtention de développements asymptotiques topologiques pour des équations elliptiques non linéaires, visant à étendre la portée applicative des méthodes de sensibilité topologique, tant en optimisation de forme qu'en traitement d'images.

Les *méthodes de développements asymptotiques topologiques*, également appelées de *gradient topologique* ou de *sensibilité topologique* ont été développées depuis les années 1990 [78, 63, 68, 81]. Elles sont appliquées en optimisation de forme [81, 44, 3, 14, 83, 58] comme en traitement d'images [20, 21, 22, 23, 24, 25, 26, 57, 15, 58].

L'idée centrale consiste à quantifier au travers de la variation d'une fonctionnelle, la variation de la solution d'une équation aux dérivées partielles, lorsque cette dernière est perturbée au voisinage d'un point donné x_0, dans un sous-domaine dont un paramètre géométrique tend vers 0. Plus précisément, soit $\Omega \subset \mathbb{R}^N$ un domaine borné. On se donne une équation aux dérivées partielles dans Ω, par exemple celle de Laplace, avec condition frontière sur $\partial\Omega$, qui admet une unique solution u_0 dans un espace fonctionnel approprié $\mathcal{F}_0$. Soit $\omega \subset \mathbb{R}^N$ un domaine borné contenant l'origine 0. Soit un point $x_0 \in \Omega$ et $\varepsilon > 0$ assez petit tel que $\omega_\varepsilon := x_0 + \varepsilon\, \omega \subset \Omega$.

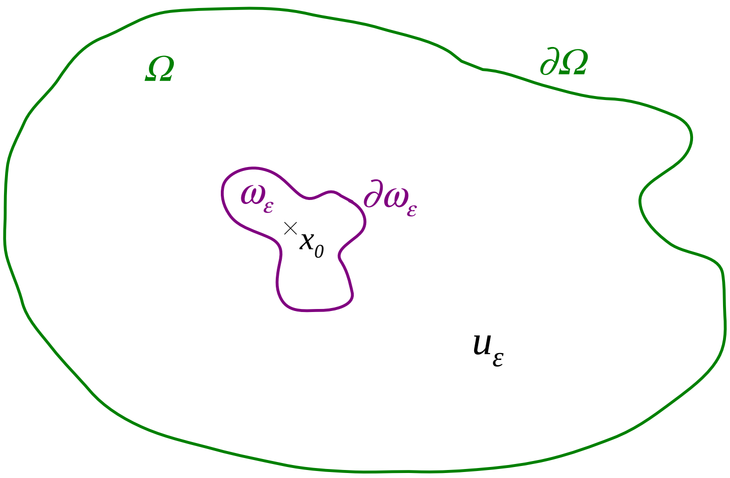

FIGURE 5.9 – Une équation perturbée dans ω_ε

On perturbe alors l'équation dans ω_ε (voir Figure 5.9), soit en changeant un coefficient dans ω_ε, par exemple une conductivité, soit en restreignant le domaine de l'équation à $\Omega_\varepsilon := \Omega \setminus \bar{\omega}_\varepsilon$ et en ajoutant une condition sur la frontière $\partial\omega_\varepsilon$. On suppose qu'à son tour l'équation perturbée admet une unique solution u_ε dans un espace fonctionnel $\mathcal{F}_\varepsilon$.

Soit $J_\varepsilon : \mathcal{F}_\varepsilon \to \mathbb{R}$ une fonctionnelle définie pour $\varepsilon \geq 0$ assez petit. Alors le *développement asymptotique topologique*, s'il existe, est de la forme

$$J_\varepsilon(u_\varepsilon) = J_0(u_0) + \rho(\varepsilon)\ g(x_0) + o(\rho(\varepsilon)), \quad \forall\, \varepsilon \geq 0 \text{ assez petit,} \tag{5.6.1}$$

où ρ est une fonction à valeurs positives telle que $\lim\limits_{\varepsilon\to 0} \rho(\varepsilon) = 0$. Le scalaire $g(x_0)$ est appelé le *gradient topologique* au point x_0. On définit ainsi la *fonction gradient topologique* $g : \Omega \to \mathbb{R}$.

Le signe et l'amplitude des valeurs prises par g dans Ω constituent les informations décisionnelles qui sont exploitées par des algorithmes dédiés, selon les tâches applicatives à réaliser.

Les développements asymptotiques topologiques ont été obtenus pour les équations de l'élasticité linéaire [44], de Helmholtz [77], de Stokes [48] et de Navier-Stokes [10].

Concernant l'équation de Laplace, les développements asymptotiques topologiques ont été établis et appliqués avec condition de Dirichlet [47, 11] comme avec condition de Neumann [44, 82, 70, 13].

D'un point de vue mathématique, la question des développements asymptotiques topologiques pour des équations elliptiques non linéaires d'ordre 2 se pose donc naturellement. Le cas des équations semilinéaires (opérateur de Laplace additionné à un terme non linéaire dépendant de la solution) a fait l'objet de deux articles [11, 50]. Au meilleur de notre connaissance, les développements asymptotiques topologiques restent à ce jour inconnus pour les équations elliptiques non linéaires, avec opérateur différentiel non linéaire, comme les équations quasilinéaires et en particulier l'équation de p-Laplace.

En outre cette question ressort d'au moins deux domaines applicatifs.

1. En optimisation de forme, le recours aux équations de l'élasticité linéaire constitue une importante limitation chaque fois que le comportement réel des structures mécaniques en jeu relève des équations non linéaires de l'élasticité. Cette question a été soulevée depuis [2] §8.

2. En traitement d'image, la détection d'objets de codimension ≥ 2, comme les points en $2D$ ou les courbes en $3D$, demeure un objectif important, par exemple en imagerie médicale. Une courbe régulière peut être localement assimilée à un segment de longueur suffisamment petite. La détection de segments en $2D$ a été étudiée dans [9] par une méthode de sensibilité topologique appliquée à l'équation de Laplace avec condition de Neumann. Mais conformément à la théorie non linéaire du potentiel, l'équation de Laplace ne peut détecter des objets que si leur codimension est < 2. Ainsi elle ne peut détecter ni des points en $2D$ ni des segments en $3D$. Pour de telles tâches il devient nécessaire de considérer l'équation de p-Laplace, où p doit être choisi strictement supérieur à la codimension des objets à détecter.

Chapitre 6

Développements asymptotiques topologiques pour des équations elliptiques quasilinéaires

Notations pour le chapitre 6

Soit $N \in \mathbb{N}$, $N \geq 2$. Soit $p \in [2, \infty)$ et q défini par $1/p + 1/q = 1$.
Quelques notations usuelles seront utilisées, comme suit :

1. le symbole $|E|$ désigne soit la norme euclidienne de E dans $\mathbb{R}^N$ quand $E \in \mathbb{R}^N$, soit la mesure N-dimensionnelle de E quand $E \subset \mathbb{R}^N$.
2. Pour tout $a > 0$, on note $B_a := \left\{x \in \mathbb{R}^N ; |x| < a\right\}$ et $B'_a := \mathbb{R}^N \setminus \overline{B}_a$.
3. S^{N-1} désigne la sphère unité dans $\mathbb{R}^N$ and A^{N-1} l'aire de sa surface.
4. I_N désigne la matrice unité N-dimensionnelle.
5. Pour tout ouvert $\mathcal{O} \subset \mathbb{R}^N$ ou $\mathcal{O} \subset \mathbb{R}$, $C_0^\infty(\mathcal{O})$ désigne l'espace des fonctions indéfiniment dérivables à support compact $\subset \mathcal{O}$ et $\mathcal{D}'(\mathcal{O})$ l'espace des distributions sur $\mathcal{O}$.
6. Le dual topologique d'un espace vectoriel normé $\mathcal{F}$ est noté $\mathcal{F}^*$.

Soit Ω un domaine borné de $\mathbb{R}^N$. On note

1. l'espace de Sobolev $W^{1,p}(\Omega) := \{u \in \mathcal{D}'(\Omega) ; u \in L^p(\Omega), \nabla u \in L^p(\Omega)\}$ muni de la norme $\|u\|_{1,p} := \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}$;
2. $\mathcal{V} := W_0^{1,p}(\Omega)$ l'adhérence de $C_0^\infty(\Omega)$ dans $W^{1,p}(\Omega)$;
3. l'espace de Hilbert $H^1(\Omega) := \{u \in \mathcal{D}'(\Omega) ; u \in L^2(\Omega), \nabla u \in L^2(\Omega)\}$ muni de la norme $\|u\|_{1,2} := \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$;
4. $\mathcal{H} := H_0^1(\Omega)$ l'adhérence de $C_0^\infty(\Omega)$ dans $H_1(\Omega)$.

6.1 Difficultés soulevées par les équations elliptiques quasilinéaires

Nous analysons les difficultés spécifiques qui apparaissent dans le processus d'obtention du développement asymptotique topologique pour une équation elliptique quasilinéaire du second ordre.

Comme point de départ, nous considérons les étapes clés mises en œuvre dans le cas de l'équation de Laplace. Dans ce cas, les états directs et adjoints sont définis par des équations elliptiques linéaires dans un même espace de Hilbert, par application du théorème de Lax-Milgram. Par identification de l'espace de Hilbert avec son dual, la mise en dualité de l'état direct avec l'état adjoint est immédiate. Les comportements asymptotiques de ces états à l'échelle 1 dans $\mathbb{R}^N$ sont bien connus dans la littérature. Ils conduisent à une distinction simple de ce qui peut être considéré comme loin de la perturbation (avec une énergie négligeable au premier ordre) par opposition à ce qui est proche de la perturbation (avec une énergie à prendre en compte au premier ordre dans le développement asymptotique). L'établissement du développement asymptotique topologique s'en suit.

Considérant maintenant l'équation de p-Laplace, $p \in (2, \infty)$, pour illustrer le cas non linéaire, nous faisons apparaître qu'une difficulté centrale réside dans la définition de la variation de l'état direct à l'échelle 1. Notant $U_0 := \nabla u_0(x_0)$, il s'agit de trouver un espace fonctionnel $\mathcal{V}(\mathbb{R}^N)$ tel que l'opérateur non linéaire $A : \mathcal{V}(\mathbb{R}^N) \to \mathcal{V}^*(\mathbb{R}^N)$ défini par

$$\langle Au, \eta \rangle := \int_{\mathbb{R}^N} \gamma \left[|U_0 + \nabla u|^{p-2} (U_0 + \nabla u) - |U_0|^{p-2} U_0 \right] .\nabla \eta, \quad \forall u, \eta \in \mathcal{V}(\mathbb{R}^N),$$

soit bien défini dans $\mathcal{V}(\mathbb{R}^N)$ et satisfasse en outre les propriétés de continuité, de coercivité et de stricte monotonie requises pour appliquer le théorème de Minty-Browder (cf. [30] Thm. V.15).

Une inégalité de type Poincaré doit bien sûr y assurer l'équivalence entre la norme et la semi-norme. Plus fondamentalement, les exigences concomitantes de bonne définition, de continuité et de coercivité de l'opérateur A font apparaître la nécessité d'une coexistence des normes L^p and L^2 du gradient dans la définition de l'espace.

En effet, il existe $C > 0$ tel que pour tous u, η, avec $\nabla u, \nabla \eta \in L^p \cap L^2(\mathbb{R}^n)$, on ait la majoration

$$|\langle Au, \eta \rangle| \leq C \left(|U_0|^{p-2} \|\nabla u\|_{L^2(\mathbb{R}^N)} \|\nabla \eta\|_{L^2(\mathbb{R}^N)} + \|\nabla u\|_{L^p(\mathbb{R}^N)}^{\frac{p}{q}} \|\nabla \eta\|_{L^p(\mathbb{R}^N)} \right).$$

Cette majoration assure la bonne définition et la continuité de A dans un espace de fonctions dont les gradients sont à la fois $L^p(\mathbb{R}^n)$ et $L^2(\mathbb{R}^n)$.

En terme de minoration, on dispose seulement de l'inégalité suivante : il existe $c > 0$ tel que pour tout u, avec $\nabla u \in L^p(\mathbb{R}^n)$, on ait

$$\langle Au, u \rangle = \int_{\mathbb{R}^N} \gamma \left[|U_0 + \nabla u|^{p-2} (U_0 + \nabla u) - |U_0|^{p-2} U_0 \right] .\nabla u \geq c \|\nabla u\|_{L^p(\mathbb{R}^N)}^p .$$

Cette minoration montre que l'opérateur A ne peut être coercif par rapport à la norme $L^2(\mathbb{R}^n)$ du gradient.

Ainsi l'analyse de cette question montre qu'en comparaison de la méthode appliquée dans le cas linéaire, la seule étape de définition de l'état direct à l'échelle 1 dans $\mathbb{R}^N$ par le théorème de Minty-Browder exige à la fois :

- de construire un espace fonctionnel, qui sera noté $\mathcal{V}(\mathbb{R}^N)$ ci-dessous, qui satisfait une inégalité de Poincaré et dont la norme permet de contrôler simultanément les normes L^p et L^2 du gradient ;

– de considérer une équation elliptique quasilinéaire conduisant à un opérateur A vérifiant une double propriété de p et 2 ellipticité. Tel n'est pas le cas de l'équation de p-Laplace.

La première exigence justifie que nous construisions l'espace de Banach réflexif $\mathcal{V}(\mathbb{R}^n)$ (et l'espace de Hilbert $\mathcal{H}(\mathbb{R}^N)$ quand $p = 2$) à la section 6.2 ci-dessous. La seconde exigence explique pourquoi nous considérons à la section 6.3 en page 151, une classe d'équations elliptiques quasilinéaires pour lesquelles l'opérateur A bénéficie d'une double propriété de p et 2 ellipticité. L'un des prix à payer sera d'avoir à estimer des quantités de la forme

$$\|\nabla \tilde{u}\|_{L^p}^p + \|\nabla \tilde{u}\|_{L^2}^2 ,$$

pour la variation $\tilde{u}$ de l'état direct aux différentes étapes d'approximation.

Outre les conditions ci-dessus assurant la bonne définition de l'état direct à l'échelle 1, et revenant à la comparaison avec la méthode mise en œuvre pour l'équation de Laplace, il subsiste que plusieurs autres étapes clés seront à adapter spécifiquement au contexte non linéaire, à savoir :

1. assurer la mise en dualité entre les variations des états direct et adjoint aux différentes étapes d'approximation. A la section 6.3.2, nous définirons l'état adjoint comme solution d'une équation adjointe linéarisée dans l'espace de Hilbert $\mathcal{H} = H_0^1(\Omega)$, puis à l'échelle 1 dans l'espace de Hilbert $\mathcal{H}(\mathbb{R}^n)$. Le schéma de dualité mis en œuvre repose sur l'injection $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$ à l'échelle ε et, à l'échelle 1, sur la «fausse injection» : pour tout $\eta \in L^\infty(\mathbb{R}^N)$, on a

$$\eta \in \mathcal{V}(\mathbb{R}^n) \Rightarrow \eta \in \mathcal{H}(\mathbb{R}^n), \qquad \text{avec} \qquad |u|_{\mathcal{H}(\mathbb{R}^n)} \leq |u|_{\mathcal{V}(\mathbb{R}^n)} .$$

2. déterminer le comportement asymptotique de la variation de l'état direct à l'échelle 1 dans $\mathbb{R}^N$. Cette fonction sera solution d'un problème non linéaire de transmission dans $\mathcal{V}(\mathbb{R}^n)$, pour lequel les techniques de convolution de la source avec une solution élémentaire ne peuvent plus s'appliquer comme dans le cas linéaire. Il faudra donc construire une sur-solution et une sous-solution puis prouver un théorème de comparaison.
3. déterminer ce qui devra être considéré comme loin de la perturbation par opposition à ce qui sera considéré comme proche de la perturbation. Cette question sera traitée aux Propositions 6.3.6 et 6.3.7.

6.2 Espaces de Sobolev à poids et quotients

Cette section étend au cas non linéaire dans le cadre d'espaces de Banach réflexifs les définitions et théorèmes exposés jusqu'ici dans le cadre d'espaces de Hilbert ([40], volume 6, chapter XI ; [8], Annexe A et [16], Appendix B). Le schéma de construction de ces espaces est classique. La principale difficulté réside dans le fait que l'espace quotient ne s'identifie plus à un sous-espace fermé de l'espace initial. La Proposition 6.2.1 permet de dépasser cette difficulté et de conclure la preuve de l'inégalité de Poincaré. Enfin la Proposition 6.2.2 apportera la double propriété de p et 2 coercivité recherchée.

On introduit les poids $w_p : \mathbb{R}^N \to \mathbb{R}$ définis par : pour tout $x \in \mathbb{R}^N$,

$$w_p(x) := \begin{cases} \left(1+|x|^2\right)^{-\frac{N+1}{2p}}, & \text{si } p < N, \\ \left(1+|x|^2\right)^{-\frac{1}{2}}, & \text{si } p > N, \\ \left(1+|x|^2\right)^{-\frac{1}{2}} \left(\log(2+|x|)\right)^{-1}, & \text{si } p = N. \end{cases}$$

On définit les espaces de Sobolev à poids suivants :

– $\mathcal{W}^w(\mathbb{R}^N) := \left\{ u \in \mathcal{D}'(\mathbb{R}^N) \ ; \ w_p u \in L^p(\mathbb{R}^N), \nabla u \in L^p(\mathbb{R}^N) \right\}$ muni de la norme

$$\|u\|_{\mathcal{W}^w(\mathbb{R}^N)} := \|w_p u\|_{L^p(\mathbb{R}^N)} + \|\nabla u\|_{L^p(\mathbb{R}^N)}, \forall u \in \mathcal{W}^w(\mathbb{R}^N);$$

– le sous-espace $\mathcal{V}^w(\mathbb{R}^N) := \left\{ u \in \mathcal{W}^w(\mathbb{R}^N) \ ; \nabla u \in L^2(\mathbb{R}^N) \right\}$ muni de la norme

$$\|u\|_{\mathcal{V}^w(\mathbb{R}^N)} := \|u\|_{\mathcal{W}^w(\mathbb{R}^N)} + \|\nabla u\|_{L^2(\mathbb{R}^N)}, \forall u \in \mathcal{V}^w(\mathbb{R}^N);$$

– et $\mathcal{H}^w(\mathbb{R}^N) := \left\{ u \in \mathcal{D}'(\mathbb{R}^N) \ ; \ w_2 u \in L^2(\mathbb{R}^N), \nabla u \in L^2(\mathbb{R}^N) \right\}$ muni du produit scalaire

$$\langle u, v \rangle_{\mathcal{H}^w(\mathbb{R}^N)} := \langle w_2 u, w_2 v \rangle_{L^2(\mathbb{R}^N)} + \langle \nabla u, \nabla v \rangle_{L^2(\mathbb{R}^N)}, \forall u, v \in \mathcal{H}^w(\mathbb{R}^N).$$

L'espace $\mathcal{W}^w(\mathbb{R}^N)$ (resp. $\mathcal{V}^w(\mathbb{R}^N)$) muni de la norme $\|.\|_{\mathcal{W}^w(\mathbb{R}^N)}$ (resp. $\|.\|_{\mathcal{V}^w(\mathbb{R}^N)}$) est un espace de Banach réflexif. L'espace $\mathcal{H}^w(\mathbb{R}^N)$ muni du produit scalaire $\langle ., . \rangle_{\mathcal{H}^w(\mathbb{R}^N)}$ est un espace de Hilbert.

Ces espaces sont ensuite quotientés par leur sous-espace fermé $\mathbb{R}$. L'espace quotient $\mathcal{W}(\mathbb{R}^N) := \mathcal{W}^w(\mathbb{R}^N)/\mathbb{R}$ est muni de la norme

$$\|[u]\|_{\mathcal{W}(\mathbb{R}^N)} := \inf_{m \in \mathbb{R}} \|u+m\|_{\mathcal{W}^w(\mathbb{R}^N)}, \quad \forall \ [u] \in \mathcal{W}(\mathbb{R}^N), \tag{6.2.1}$$

où $u \in \mathcal{W}^w(\mathbb{R}^N)$ désigne un élément quelconque de la classe $[u]$. Le quotient $\mathcal{W}(\mathbb{R}^N)$ reste un espace de Banach réflexif.

L'espace quotient $\mathcal{V}(\mathbb{R}^N) := \mathcal{V}^w(\mathbb{R}^N)/\mathbb{R}$ (resp. $\mathcal{H}(\mathbb{R}^N) := \mathcal{H}^w(\mathbb{R}^N)/\mathbb{R}$) est défini de façon similaire et reste un espace de Banach réflexif (resp. un espace de Hilbert).

Pour une classe donnée $[u] \in \mathcal{W}(\mathbb{R}^N)$, le problème de minimisation posé dans la définition (6.2.1) admet un unique minimiseur $v^* \in u + \mathbb{R}$ caractérisé par l'équation d'Euler-Lagrange :

$$\int_{\mathbb{R}^N} w_p^p \, |v^*|^{p-2} \, v^* = 0.$$

Contrairement au cas hilbertien ($p = 2$), le lieu des minimiseurs quand $[u]$ décrit $\mathcal{W}(\mathbb{R}^N)$, soit

$$\mathcal{M}_p := \left\{ v \in \mathcal{W}^w(\mathbb{R}^N); \int_{\mathbb{R}^N} w_p^p \, |v|^{p-2} \, v = 0 \right\}, \tag{6.2.2}$$

n'est pas un sous-espace fermé de l'espace initial. Généralisant la relation $\mathbb{R}^\perp \cap \mathbb{R} = \{0\}$ valable dans le cas hilbertien, on montre toutefois que

Proposition 6.2.1. Soit $(v_l)_{l\in\mathbb{N}}$ une suite $\subset \mathcal{M}_p$ et $m \in \mathbb{R}$ tels que

$$\lim_{l\to+\infty} \int_{\mathbb{R}^N} w_p^p \, |v_l + m|^{p-2} \, (v_l + m) \; = \; 0. \tag{6.2.3}$$

Alors $m = 0$.

En termes géométriques, la condition (6.2.3) exprime que $v_l + m$ devient asymptotiquement un minimiseur pour la classe $[v_l + m] = [v_l]$. Puisque $v_l \in \mathcal{M}_p$ est déjà un minimiseur pour cette classe, l'unicité du minimiseur conduit en quelque sorte à $m = 0$.

Cette propriété permet de conclure la preuve par l'absurde du théorème de Poincaré désiré dans $\mathcal{W}(\mathbb{R}^N)$.

Théorème 6.2.1. Il existe $c > 0$ tel que

$$\|[u]\|_{\mathcal{W}(\mathbb{R}^N)} \leq c \, \|\nabla u\|_{L^p(\mathbb{R}^N)}\,, \quad \forall\; [u] \in \mathcal{W}(\mathbb{R}^N),$$

où u désigne un élément quelconque de la classe $[u]$.

Il en découle l'équivalence de la norme et de la semi-norme dans $\mathcal{W}(\mathbb{R}^N)$. Les inégalités de Poincaré et les équivalences entre normes et semi-normes s'en déduisent immédiatement dans l'espace de Banach $\mathcal{V}(\mathbb{R}^N)$ comme dans l'espace de Hilbert $\mathcal{H}(\mathbb{R}^N)$.

On établit enfin par la proposition suivante, la propriété recherchée de double p et 2 coercivité dans $\mathcal{V}(\mathbb{R}^N)$.

Proposition 6.2.2. On a

$$\lim_{\|[u]\|_{\mathcal{V}(\mathbb{R}^N)}\to\infty} \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{\|[u]\|_{\mathcal{V}(\mathbb{R}^N)}} \; = \; +\infty,$$

où u désigne un élément quelconque de la classe $[u]$.

6.3 Asymptotiques topologiques pour des EDPs quasilinéaires

Le résultat principal de cette section et de ce chapitre réside dans l'établissement du développement asymptotique topologique pour une classe d'équations elliptiques quasilinéaires, résultat énoncé au Théorème 6.3.1.

On définit d'abord le type considéré d'équations à perturber. On rappelle que dans ce chapitre, $N \geq 2$, $p \in [2, \infty)$ et Ω est un domaine borné et régulier de $\mathbb{R}^N$. Soit $x_0 \in \Omega$. On va perturber la conductivité dans un sous-domaine $\omega_\varepsilon := x_0 + \varepsilon\omega$, où ω est un domaine borné régulier tel que $0 \in \omega$. On se donne donc une fonction de conductivité $\gamma_\varepsilon : \Omega \to \mathbb{R}$ définie par

$$\gamma_\varepsilon := \gamma_0 \text{ dans } \Omega \setminus \omega_\varepsilon \;\text{ et }\; \gamma_\varepsilon := \gamma_1 \text{ dans } \omega_\varepsilon,$$

où $\gamma_0 \neq \gamma_1$ sont deux réels strictement positifs.

Soit un potentiel $W \in C^{2,\alpha}(\mathbb{R}^N, \mathbb{R})$ et une source $f \in C^{0,\alpha}(\Omega)$, avec un support $spt(f) \subset\subset \Omega$. On prendra $x_0 \in \Omega \setminus spt(f)$.

Soit $\mathcal{V} := W_0^{1,p}(\Omega)$. En raison des hypothèses qui seront faites sur le potentiel W, la fonctionnelle

$$\mathcal{W}_\varepsilon : \eta \in \mathcal{V} \mapsto \int_\Omega \gamma_\varepsilon W(\nabla\eta) \; - \; \int_\Omega f\eta$$

sera Fréchet différentiable, strictement convexe et coercive dans $\mathcal{V}$. On définit donc de façon unique l'état direct perturbé par

$$\{u_\varepsilon\} = \operatorname*{argmin}_{\eta\in\mathcal{V}} \mathcal{W}_\varepsilon(\eta).$$

De façon équivalente, l'état direct perturbé u_ε est entièrement déterminé par l'équation d'Euler-Lagrange :

$$\text{trouver } u_\varepsilon \in \mathcal{V} \text{ tel que } \int_\Omega \gamma_\varepsilon T(\nabla u_\varepsilon).\nabla\eta = \int_\Omega f\eta, \quad \forall \eta \in \mathcal{V},$$

dont la forme forte est :

$$\text{trouver } u_\varepsilon \in W^{1,p}(\Omega) \text{ tel que} \begin{cases} -\operatorname{div}(\gamma_\varepsilon T(\nabla u_\varepsilon)) = f & \text{dans } \Omega, \\ u_\varepsilon = 0 & \text{sur } \partial\Omega. \end{cases}$$

On précise maintenant la classe considérée de potentiels W. On note $T := \nabla W : \mathbb{R}^N \to \mathbb{R}^N$ le champ de gradient de W. A l'ordre de dérivation suivant, pour tout $\varphi \in \mathbb{R}^N$, on définit l'opérateur S_φ pour caractériser la non linéarité du champ T au point φ. On pose

$$S_\varphi(\psi) := T(\varphi+\psi) - T(\varphi) - DT(\varphi).\psi, \quad \forall\, \psi \in \mathbb{R}^N.$$

Cet opérateur S servira à rendre compte de la non linéarité de l'équation dans la formule du gradient topologique. Lorsque le potentiel W est quadratique, l'opérateur S est identiquement nul.

L'archétype de potentiel dans le classe considérée est de la forme

$$W_a \quad : \quad \varphi \in \mathbb{R}^N \mapsto \frac{1}{p}\left(a^2 + |\varphi|^2\right)^{p/2}, \tag{6.3.1}$$

pour un certain $a > 0$.

On montre qu'il vérifie la double condition de p et 2 ellipticité attendue grâce à la propriété suivante :

Proposition 6.3.1. Soit $a > 0$ et $p \in [2, \infty)$. Alors il existe $c > 0$ tel que

$$\left[(a^2 + |\varphi+\psi|^2)^{\frac{p-2}{2}}(\varphi+\psi) - (a^2+|\varphi|^2)^{\frac{p-2}{2}}\varphi\right].\psi \geq c\,(|\psi|^p + |\psi|^2), \quad \forall\, \varphi, \psi \in \mathbb{R}^N.$$

Naturellement, on peut vérifier que la 2-ellipticité s'évanouit lorsque $a \to 0$.

Plus généralement, on dira qu'un potentiel W fait partie de la classe considérée, s'il satisfait un ensemble de conditions, que l'on peut résumer comme suit :

- une condition de régularité sur W ;
- des conditions classiques de croissance du potentiel, du champ de gradient et de la hessienne, qui garantissent en particulier la stricte convexité du potentiel ;
- la double condition de p et 2 ellipticité de la variation du champ de gradient ;

– deux conditions relatives à la croissance de l'opérateur S.

Plus précisément et de façon axiomatique, on définit la classe considérée de potentiels W par l'Hypothèse 6.3.2 ci-dessous.

Hypothèse 6.3.2. Le potentiel W satisfait les conditions suivantes :

1. $W \in C^{2,\alpha}(\mathbb{R}^N, \mathbb{R})$ pour un certain $\alpha > 0$.
2. Il existe $b_0 > a_0 > 0$ tels que
$$a_0 \, |\varphi|^p \le W(\varphi) \le b_0(1 + |\varphi|^p), \quad \forall \, \varphi \in \mathbb{R}^N.$$
3. Il existe $a_1 > 0$ tel que
$$|T(\varphi)| \le a_1 \, |\varphi| \, (1 + |\varphi|^{p-2}), \quad \forall \, \varphi \in \mathbb{R}^N.$$
4. Il existe $0 < c < C$ tels que
$$c(1 + |\varphi|^2)^{\frac{p-2}{2}} \, |\psi|^2 \le DT(\varphi)\psi.\psi \le C(1 + |\varphi|^2)^{\frac{p-2}{2}} \, |\psi|^2 \,, \quad \forall \, \varphi, \psi \in \mathbb{R}^N.$$
5. Il existe $c > 0$ tel que
$$(T(\varphi + \psi) - T(\varphi)) \, .\psi \ge c(|\psi|^p + |\psi|^2), \quad \forall \, \varphi, \psi \in \mathbb{R}^N.$$
6. Il existe $C > 0$ tel que
$$|T(\varphi + \psi) - T(\varphi)| \le C \, |\psi| \left[1 + |\varphi|^{p-2} + |\psi|^{p-2} \right], \quad \forall \, \varphi, \psi \in \mathbb{R}^N.$$
7. Soit $M > 0$. Alors il existe $c_0 = c_0(M, p) \ge 0$ et $c_{p-3} = c_{p-3}(p) \ge 0$ tels que
$$|S_\varphi(\psi_2) - S_\varphi(\psi_1)| \le |\psi_2 - \psi_1| \, (|\psi_1| + |\psi_2|) \left[c_0 + c_{p-3} \, (|\psi_1| + |\psi_2|)^{p-3} \right],$$
$$\forall \varphi \in B(0, M), \; \forall \psi_1, \psi_2 \in \mathbb{R}^N.$$
De plus pour tout $M > 0$, les constantes c_0 et c_{p-3} satisfont les conditions suivantes :
$$\begin{cases} c_{p-3} = 0, & \forall p \in [2, 3] \,, \\ c_0 = 0, & \text{si } p = 2. \end{cases}$$
8. Soit $M > 0$. Alors il existe $d_0 = d_0(M, p) \ge 0$ et $d_{p-4} = d_{p-4}(p) \ge 0$ tels que
$$|S_{\varphi_2}(\psi) - S_{\varphi_1}(\psi)| \le |\varphi_2 - \varphi_1| \, |\psi|^2 \left[d_0 + d_{p-4} \, |\psi|^{p-4} \right], \quad \forall \varphi_1, \varphi_2 \in B(0, M), \, \forall \psi \in \mathbb{R}^N.$$
De plus pour tout $M > 0$, les constantes d_0 et d_{p-4} satisfont les conditions suivantes :
$$\begin{cases} d_{p-4} = 0, & \forall p \in [2, 4] \,, \\ d_0 = 0, & \text{si } p = 2. \end{cases}$$

L'objet de ces deux dernières conditions est de contrôler les quantités de la forme :

– $S_\varphi(\psi_2) - S_\varphi(\psi_1)$, i.e. la variation de S_φ entre deux points ψ_1 et ψ_2 ;
– et $S_{\varphi_2}(\psi) - S_{\varphi_1}(\psi)$, i.e. la différence vue d'un même point ψ entre l'opérateur S_{φ_1} et l'opérateur S_{φ_2}.

Il sera fait un usage intensif de ces deux majorations dans l'établissement du développement asymptotique topologique.

Enfin, on considère une classe de fonctionnelles, dont l'archétype est la compliance

$$u \in \mathcal{H} \mapsto \int_\Omega fu.$$

Plus généralement, une famille de fonctionnelles $\mathcal{J}_\varepsilon : \mathcal{H} \to \mathbb{R}$ sera admise dans la classe s'il existe un développement asymptotique de la forme

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_0(u_0) + \langle G, u_\varepsilon - u_0 \rangle + \delta_2\, \varepsilon^N + R(\varepsilon), \tag{6.3.2}$$

où

1. G est une forme linéaire continue sur $\mathcal{H}$;
2. $\delta_2 \in \mathbb{R}$;
3. le reste $R(\varepsilon)$ est
   (a) soit de la forme
   $$R(\varepsilon) = o\left(\|u_\varepsilon - u_0\|^2_{\mathcal{H}}\right), \tag{6.3.3}$$
   (b) soit de la forme
   $$R(\varepsilon) = O\left(\int_{\Omega \setminus B(0,\tilde{\alpha}\varepsilon^{\tilde{r}})} |\nabla(u_\varepsilon - u_0)|^p + |\nabla(u_\varepsilon - u_0)|^2\right), \tag{6.3.4}$$
   pour un certain $\tilde{\alpha} > 0$ et un certain $\tilde{r} \in (0,1)$.

Nous pouvons maintenant énoncer le développement asymptotique de $\mathcal{J}_\varepsilon(u_\varepsilon)$. On note :

– u_0 l'état direct non perturbé, i.e. quand $\varepsilon = 0$;
– $U_0 := \nabla u_0(x_0)$;
– H la variation de l'état direct à l'échelle 1 dans $\mathbb{R}^N$;
– v_0 l'état adjoint non perturbé ;
– $V_0 := \nabla v_0(x_0)$;
– K la variation de l'état adjoint à l'échelle 1 dans $\mathbb{R}^N$;
– γ la fonction de conductivité à l'échelle 1 ;
– $\mathcal{P}$ est un tenseur de polarisation, qui ne dépend que de ω, de $DT(U_0)$ et du ratio γ_1/γ_0.

Théorème 6.3.1. On suppose que :

– le potentiel W fait partie de la classe définie par l'Hypothèse 6.3.2 ;
– la famille de fonctionnelles $(\mathcal{J}_\varepsilon)$ vérifie un développement asymptotique du type (6.3.2) ;
– $u_0 \in L^\infty(\Omega)$;
– $v_0 \in L^\infty(\Omega)$, $\nabla v_0 \in L^\infty(\Omega)$, et v_0 et ∇v_0 sont Hölder continus au point x_0 ;

– le comportement asymptotique à l'infini de H satisfait l'Hypothèse 6.3.5.

Alors pour tout $\varepsilon > 0$ assez petit, on a

$$\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) = \varepsilon^N g(x_0) + o(\varepsilon^N), \tag{6.3.5}$$

où

$$\begin{aligned} g(x_0) \quad :=& \quad T(U_0)^T \mathcal{P} V_0 + \delta_2 &(6.3.6)\\ &+ \int_{\mathbb{R}^N} \gamma S_{U_0}(\nabla H).\,(V_0 + \nabla K)\,. &(6.3.7)\end{aligned}$$

Ainsi deux termes apparaissent dans le gradient topologique.

- Dans le cas linéaire, où $S_{U_0} = 0$, le gradient topologique $g(x_0)$ se limite au premier terme (6.3.6). Il peut donc être estimé en calculant les champs de gradient de u_0 et de v_0 et le tenseur de polarisation $\mathcal{P}$, qui cependant varie avec $DT(U_0)$.
- Le terme (6.3.7) est publié ici pour la première fois. Il rend compte de la non-linéarité du champ de gradient T en U_0. En vue des applications, le coût de son calcul sera déterminant.

Il faut par ailleurs souligner que les hypothèses de régularité faites dans l'énoncé du Théorème 6.3.1 sont beaucoup moins contraignantes en pratique qu'il n'y paraît à première vue. En effet :

- Dans le cas central, où $\omega = B(0,1)$, $W = W_a$ pour un certain $a > 0$ et $\gamma_1 < \gamma_0$, on définit la valeur limite supérieure

  $$\bar{p} := 2 + \left(1 + \frac{a^2}{|U_0|^2}\right) \frac{N}{N-2},$$

  avec la convention $\bar{p} = +\infty$ quand $N = 2$.
  Si $p < \bar{p}$, alors aucune hypothèse dans l'énoncé du Théorème 6.3.1 n'est nécessaire quant au comportement asymptotique de H, puisque ce comportement est dès lors garanti par le Théorème 6.3.2 énoncé en page 156.
- L'hypothèse $u_0 \in L^\infty(\Omega)$ est théoriquement nécessaire pour prouver la régularité $C^{1,\beta}(\overline{\Omega})$ de u_0. En pratique, cette condition peut être considérée comme acquise.
- Quand G est assez régulier, on montre que v_0 est $C^{1,\tilde{\beta}}(\overline{\Omega})$. Dans ce cas, aucune hypothèse de régularité de v_0 n'est nécessaire dans l'énoncé du Théorème 6.3.1.

Pour prouver le Théorème 6.3.1, il est nécessaire d'étudier :

1. la variation de l'état direct ;
2. la variation de l'état adjoint ;
3. le développement asymptotique de $\mathcal{J}_\varepsilon(u_\varepsilon)$.

Les points clés résident dans :

1. la détermination du comportement asymptotique de la variation H de l'état direct à l'échelle 1 dans $\mathbb{R}^N$;
2. la détermination de ce qui doit être considéré comme loin versus proche de la perturbation ;
3. la mise en dualité des états direct et adjoint, aux différentes étapes d'approximation.

6.3.1 Variations de l'état direct

Après l'étude de la régularité de u_0, on étudie la variation $\tilde{u}_\varepsilon := u_\varepsilon - u_0$. On approche tout d'abord celle-ci par une variation h_ε, en approximant dans les équations, le champ ∇u_0 défini dans Ω par le champ uniforme U_0. Puis, en divisant les distances par ε, on définit la variation H de l'état direct à l'échelle 1. La double p et 2 ellipticité ainsi que la propriété de coercivité établie dans $\mathcal{V}(\mathbb{R}^N)$ à la Proposition 6.2.2 permettent d'appliquer le théorème de Minty-Browder et d'obtenir la

Proposition 6.3.3. Il existe une unique fonction $H \in \mathcal{V}(\mathbb{R}^N)$ telle que

$$\int_{\mathbb{R}^N} \gamma \left[T(U_0 + \nabla H) - T(U_0)\right].\nabla\eta + (\gamma_1 - \gamma_0)\int_\omega T(U_0).\nabla\eta = 0, \quad \forall \eta \in \mathcal{V}(\mathbb{R}^N). \quad (6.3.8)$$

L'enjeu est maintenant d'estimer le comportement asymptotique des variations de l'état direct. Pour $\varepsilon > 0$, on pose $H_\varepsilon(x) := \varepsilon H(\varepsilon^{-1}x),\ \forall x \in \Omega$. La double ellipticité conduit immédiatement à

Lemme 6.3.4.

$$\|\nabla \tilde{u}_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla \tilde{u}_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N), \quad (6.3.9)$$

$$\|\nabla h_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N), \quad (6.3.10)$$

$$\|\nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla H_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N). \quad (6.3.11)$$

L'obtention d'estimations plus précises, et notamment la mise en évidence de ce qui doit être considéré comme loin versus proche de la perturbation, requiert d'établir le comportement asymptotique à l'infini de la fonction H.

La fonction H est solution d'une équation de transmission non linéaire dans $\mathbb{R}^N$. Le comportement asymptotique d'une telle solution n'est pas connu dans la littérature.

Dans le cas le plus directement applicable, dans lequel le sous-domaine ω est la boule unité $B(0,1)$, où le potentiel W est de la forme W_a, pour un certain $a > 0$ et où $\gamma_1 < \gamma_0$, on peut expliciter une sur-solution de l'équation (6.3.8) dans le demi-espace $\left\{x \in \mathbb{R}^N; U_0.\ x \geq 0\right\}$. La fonction nulle est par ailleurs sous-solution dans le même demi-espace. Prouvant un théorème de comparaison et notant

$$\bar{p} := 2 + \left(1 + \frac{a^2}{|U_0|^2}\right)\frac{N}{N-2}, \quad (6.3.12)$$

avec la convention $\bar{p} = +\infty$ quand $N = 2$, on obtient

Théorème 6.3.2. On suppose que $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ et $W = W_a$ pour un certain $a > 0$. Si $p \in [2, \bar{p})$, alors il existe un élément $\tilde{H}$ de la classe $H \in \mathcal{V}(\mathbb{R}^N)$ et $\tau > \frac{N}{2} - 1$ tels que

$$\tilde{H}(y) = O\left(|y|^{-\tau}\right) \quad \text{quand} \quad |y| \to +\infty. \quad (6.3.13)$$

De plus

$$H \in L^\infty(\mathbb{R}^N). \quad (6.3.14)$$

En conséquence, on fait pour le cas général l'hypothèse suivante.

Hypothèse 6.3.5. On suppose que :

1. il existe un élément $\tilde{H}$ de la classe $H \in \mathcal{V}(\mathbb{R}^N)$ et $\tau > \frac{N}{2} - 1$ tels que

$$\tilde{H}(y) = O\left(|y|^{-\tau}\right) \quad \text{quand} \quad |y| \to +\infty \tag{6.3.15}$$

2. et

$$H \in L^\infty(\mathbb{R}^N). \tag{6.3.16}$$

En particulier, il s'ensuit que $H \in \mathcal{H}(\mathbb{R}^N)$. On montre alors la série d'estimations suivantes.

Proposition 6.3.6. On a

$$\begin{aligned}
\|\nabla h_\varepsilon - \nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^2_{L^2(\Omega)} &= o(\varepsilon^N), & (6.3.17)\\
\forall \alpha > 0,\ \forall r \in (0,1),\ \int_{\Omega \setminus B(0,\alpha\varepsilon^r)} |\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2 &= o(\varepsilon^N), & (6.3.18)\\
\int_\Omega |\nabla u_0 - U_0| \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) &= o(\varepsilon^N), & (6.3.19)\\
\forall\ p \in (4,\infty), \int_\Omega |\nabla u_0 - U_0|\, |\nabla h_\varepsilon|^{p-2} &= o(\varepsilon^N), & (6.3.20)\\
\forall\ p \in (3,\infty), \int_\Omega |\nabla u_0 - U_0|\, |\nabla h_\varepsilon|^{p-1} &= o(\varepsilon^N), & (6.3.21)\\
\int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right) &= o(\varepsilon^N), & (6.3.22)\\
\forall\ p \in (3,\infty), \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right)^{p-2} &= o(\varepsilon^N). & (6.3.23)
\end{aligned}$$

Les estimations précédentes permettent alors d'obtenir la

Proposition 6.3.7. On a

$$\begin{aligned}
\|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|^2_{L^2(\Omega)} &= o(e^N), & (6.3.24)\\
\int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right) &= o(\varepsilon^N), & (6.3.25)\\
\forall\ p \in (3,\infty), \int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right)^{p-2} &= o(\varepsilon^N), & (6.3.26)\\
\forall \alpha > 0,\ \forall\ r \in (0,1),\ \int_{\Omega \setminus B(0,\alpha\varepsilon^r)} |\nabla \tilde{u}_\varepsilon|^p + |\nabla \tilde{u}_\varepsilon|^2 &= o(\varepsilon^N). & (6.3.27)
\end{aligned}$$

Les estimations établies aux Propositions 6.3.6 et 6.3.7 seront nécessaires pour prouver le développement topologique asymptotique. Les estimations (6.3.18) et (6.3.27) répondent à la question de savoir ce qui doit considéré comme loin de la perturbation. L'énergie est négligeable au premier ordre en ε^N, en dehors d'une boule de rayon $\alpha\varepsilon^r$, pour tout $r \in (0,1)$. Ce rayon $\alpha\varepsilon^r$ est directement déterminé par le comportement asymptotique de H. Quand $\varepsilon \to 0$, la boule $B(x_0, \alpha\varepsilon^r)$ voit son rayon tendre vers 0. En revanche, à l'échelle de la perturbation ω_ε, la frontière de la boule s'éloigne indéfiniment de la perturbation.

6.3.2 Variations de l'état adjoint

L'état adjoint est défini dans l'espace de Hilbert $\mathcal{H} = H_0^1(\Omega)$. Le théorème de Lax-Milgram assure l'existence et l'unicité de $v_\varepsilon \in \mathcal{H}$ tel que

$$\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla v_\varepsilon.\nabla\eta = -\langle G, \eta\rangle\,, \quad \forall \eta \in \mathcal{H}. \tag{6.3.28}$$

Après l'étude de la régularité de v_0, on étudie la variation $\tilde{v}_\varepsilon := v_\varepsilon - v_0$. On approche tout d'abord celle-ci par une variation k_ε, en approximant dans les équations, le champ ∇v_0 défini dans Ω par le champ uniforme V_0. Puis, divisant les distances par ε, on définit la variation K de l'état adjoint à l'échelle 1, en appliquant le théorème de Lax-Milgram dans l'espace de Hilbert $\mathcal{H}(\mathbb{R}^N)$. On obtient

Lemme 6.3.8. Il existe une unique fonction $K \in \mathcal{H}(\mathbb{R}^N)$ telle que

$$\int_{\mathbb{R}^N} \gamma DT(U_0)\nabla K.\nabla\eta = -(\gamma_1 - \gamma_0)\int_\omega DT(U_0)V_0.\nabla\eta, \quad \forall\, \eta \in \mathcal{H}(\mathbb{R}^N). \tag{6.3.29}$$

La fonction K étant solution d'un problème linéaire de transmission dans $\mathbb{R}^N$, l'étude de son comportement asymptotique est classique.

Proposition 6.3.9. Il existe un élément $\tilde{K}$ de la classe $K \in \mathcal{H}(\mathbb{R}^N)$ tel que

$$\tilde{K}(y) = O\left(|y|^{1-N}\right) \quad \text{quand} \quad |y| \to +\infty, \tag{6.3.30}$$

$$\nabla K(y) = O\left(|y|^{-N}\right) \quad \text{quand} \quad |y| \to +\infty. \tag{6.3.31}$$

De plus $K \in \mathcal{V}(\mathbb{R}^N)$.

Pour $\varepsilon > 0$, on pose $K_\varepsilon(x) := \varepsilon\tilde{K}(\varepsilon^{-1}x),\ \forall x \in \Omega$. On obtient :

Lemme 6.3.10.

$$\|\nabla k_\varepsilon - \nabla K_\varepsilon\|^2_{L^2(\Omega)} = o(\varepsilon^N), \tag{6.3.32}$$

$$\forall \alpha > 0, \forall r \in (0,1), \quad \int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon|^2 = o(\varepsilon^N), \tag{6.3.33}$$

$$\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|^2_{L^2(\Omega)} = o(\varepsilon^N). \tag{6.3.34}$$

Les estimations du Lemme 6.3.10 seront nécessaires pour obtenir le développement asymptotique topologique.

6.3.3 Développement asymptotique topologique

On note $j(\varepsilon) := \mathcal{J}_\varepsilon(u_\varepsilon),\ \forall \varepsilon \geq 0$ assez petit. Du fait de l'hypothèse (6.3.2) faite sur la fonctionnelle, il vient

$$j(\varepsilon) - j(0) = \langle G, \tilde{u}_\varepsilon\rangle + \delta_2\, \varepsilon^N + o\left(\varepsilon^N\right).$$

La mise en dualité de l'état adjoint et de l'état direct est essentielle pour poursuivre le calcul. Regardant l'état direct $u_\varepsilon \in \mathcal{V} \subset \mathcal{H}$ comme fonction test de l'état adjoint

v_ε puis l'état adjoint non perturbée v_0 comme fonction test pour l'état direct u_ε, on obtient

$$j(\varepsilon) - j(0) = j_1(\varepsilon) + j_2(\varepsilon) + \delta_2\ \varepsilon^N + o(\varepsilon^N), \tag{6.3.35}$$

avec

$$j_1(\varepsilon) := (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} T(\nabla u_0).\nabla v_\varepsilon \tag{6.3.36}$$

et

$$j_2(\varepsilon) := \int_\Omega \gamma_\varepsilon S_{\nabla u_0}(\nabla \tilde{u}_\varepsilon).\nabla v_0 + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} \left[DT(\nabla u_0)\nabla v_0.\nabla \tilde{u}_\varepsilon - T(\nabla u_0).\nabla \tilde{v}_\varepsilon\right]. \tag{6.3.37}$$

Nous verrons que le terme $j_2(\varepsilon)$, qui fait apparaître l'opérateur S de non-linéarité, conduira au terme non linéaire du gradient topologique.

L'étude du terme $j_1(\varepsilon)$ est classique. Passant à l'échelle 1, on introduit

$$\begin{aligned} J_1 &:= (\gamma_1 - \gamma_0) \int_\omega T(U_0).(V_0 + \nabla K) = (\gamma_1 - \gamma_0)\ T(U_0). \left[|\omega|\, V_0 + \int_{\partial\omega} K n_{out}\right], \\ &= T(U_0).\,(\mathcal{P} V_0) = T(U_0)^T \mathcal{P} V_0 \end{aligned} \tag{6.3.38}$$

où

$$\mathcal{P} := \mathcal{P}(\omega, DT(U_0), \gamma_1/\gamma_0),$$

est une matrice de polarisation [74, 34, 5, 12].

On établit en deux étapes l'estimation

$$j_1(\varepsilon) - \varepsilon^N J_1 = o(\varepsilon^N).$$

D'où

Proposition 6.3.11.

$$j_1(\varepsilon) = \varepsilon^N\ T(U_0)^T \mathcal{P} V_0 +\ o\left(\varepsilon^N\right). \tag{6.3.39}$$

L'étude du terme $j_2(\varepsilon)$ est beaucoup plus longue et technique. Passant à l'échelle 1, on introduit

$$J_2 := \int_{\mathbb{R}^N} \gamma S_{U_0}(\nabla H).V_0 + (\gamma_1 - \gamma_0) \int_\omega \left[DT(U_0)V_0.\nabla H - T(U_0).\nabla K\right].$$

Le cadre de dualité mis en place intervient à nouveau. Comme $H \in \mathcal{H}(\mathbb{R}^N)$, la fonction H est fonction test pour K. De même, comme $K \in \mathcal{V}(\mathbb{R}^N)$, la fonction K est fonction test pour H. Ceci permet de simplifier l'expression de J_2 sous la forme

$$J_2 \ = \ \int_{\mathbb{R}^N} \gamma S_{U_0}(\nabla H).(V_0 + \nabla K).$$

On prouve d'abord le

Lemme 6.3.12. On a

$$\begin{aligned}
\int_\Omega |\nabla v_0 - V_0| \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) &= o(\varepsilon^N), & (6.3.40)\\
\forall\, p \in (3,\infty), \int_\Omega |\nabla v_0 - V_0|\, |\nabla h_\varepsilon|^{p-1} &= o(\varepsilon^N). & (6.3.41)
\end{aligned}$$

Puis on démontre en deux étapes d'approximation la

Proposition 6.3.13.

$$j_2(\varepsilon) = \varepsilon^N \left(\int_{\mathbb{R}^N} \gamma S_{U_0}(\nabla H).\,(V_0 + \nabla K)\right) + o\left(\varepsilon^N\right). \tag{6.3.42}$$

Enfin, additionnant dans la relation(6.3.35), l'estimation de $j_1(\varepsilon)$ donnée par (6.3.39) et celle de $j_2(\varepsilon)$ donnée par (6.3.42), on termine la preuve du développement asymptotique topologique annoncé dans le Théorème 6.3.1 en page 154.

6.4 Conclusions

Dans ce chapitre 6, nous avons d'abord analysé les difficultés spécifiques qui apparaissent dans le processus d'obtention du développement asymptotique topologique pour une équation elliptique quasilinéaire, par comparaison avec les étapes mises en œuvre pour une équation linéaire.

Définir la variation de l'état direct à l'échelle 1 dans $\mathbb{R}^N$ conduit à vouloir appliquer le théorème de Minty-Browder à un opérateur non linéaire spécifique, qui dépend directement de l'équation quasilinéaire considérée. Cette étape fait apparaître une nécessaire coexistence des normes L^p et L^2 du gradient, et plus précisément elle requiert

– de se placer dans un espace fonctionnel dont la norme donne le contrôle sur les normes L^p et L^2 du gradient et satisfait une inégalité de Poincaré ;
– de considérer une équation quasilinéaire elliptique telle que l'opérateur non linéaire en résultant satisfasse une propriété de double p- and 2- ellipticité, ce qui n'est pas le cas de l'équation de p-Laplace.

La première condition justifie la construction de l'espace de Sobolev à poids et quotienté $\mathcal{V}(\mathbb{R}^n)$ et de l'espace de Hilbert à poids et quotienté $\mathcal{H}(\mathbb{R}^N)$ à la section 6.2. La seconde explique le choix de la classe d'équations quasilinéaires fait en section 6.3.

Outre la bonne définition de la variation de l'état direct à l'échelle 1, plusieurs autres composantes de la méthode linéaire ont dû être adaptées au cas non linéaire. En particulier, il s'est agi :

1. d'établir le comportement asymptotique de la variation de l'état direct à l'échelle 1, celle-ci étant maintenant solution d'un problème de transmission non linéaire dans $\mathbb{R}^N$;
2. de déterminer en fonction de ce dernier comportement, ce qui doit être considéré comme loin de la perturbation, donc négligeable dans le développement asymptotique, par opposition à ce qui doit être considéré comme proche et donc intervenir dans le développement ;
3. de mettre en dualité l'état direct et l'état adjoint aux différentes étapes d'approximation.

Il en résulte notre principale contribution énoncée au Théorème 6.3.1 qui fournit le développement asymptotique topologique pour la classe considérée d'équations elliptiques quasilinéaires.

Les travaux de recherche peuvent maintenant se poursuivre dans plusieurs directions, comme par exemple

- l'obtention de développements asymptotiques topologiques pour des classes plus larges d'équations quasilinéaires, comme par exemple l'équation de p-Laplace ;
- l'obtention du développement asymptotique topologique pour les équations non linéaires de l'élasticité ;
- l'évaluation, en vue des applications, du coût du calcul du gradient topologique mis en évidence au Théorème 6.3.1.

Chapitre 7

Estimations et développements asymptotiques pour des p-capacités de condensateurs. Le cas anisotrope du segment.

Notations pour le chapitre 7

Soit $p \in (1, +\infty)$. Soit $N \in \mathbb{N}$, $N \geq 1$. Quand un obstacle en forme de segment est étudié, i.e. à la section 7.3, on suppose $N \geq 2$. Soit un domaine borné $\Omega \subset \mathbb{R}^N$ et un compact $K \subset \Omega$.

Quelques notations usuelles seront utilisées, comme suit :

1. le symbole $|E|$ désigne soit la norme euclidienne de E dans $\mathbb{R}^N$ quand $E \in \mathbb{R}^N$, soit la mesure N-dimensionnelle de E quand $E \subset \mathbb{R}^N$;
2. S^{N-1} désigne la sphère unité dans $\mathbb{R}^N$ et A^{N-1} l'aire de sa surface ;
3. $C_0^\infty(\Omega)$ désigne l'espace des fonctions indéfiniment dérivables à support compact dans Ω et $\mathcal{D}'(\Omega)$ l'espace des distributions sur Ω ;
4. l'espace de Sobolev $W^{1,p}(\Omega) := \{u \in \mathcal{D}'(\Omega); u \in L^p(\Omega), \nabla u \in L^p(\Omega)\}$ muni de la norme $\|u\|_{1,p} := \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}$;
5. $W_0^{1,p}(\Omega)$ désigne l'adhérence de $C_0^\infty(\Omega)$ dans $W^{1,p}(\Omega)$;
6. enfin on pose $\beta := (p - N)/(p - 1) \in (-\infty, 1]$. Il est commode de souvenir que $p > N \Leftrightarrow \beta > 0$, et que $\beta < 1$, pour tout $N \geq 2$.

7.1 Introduction et objectifs

La notion de capacité trouve son origine dans l'étude de la physique des condensateurs. Elle s'est depuis très largement développée en mathématiques, en théorie linéaire puis non linéaire du potentiel. Différentes définitions de la notion de capacité ont été étudiées. La notion de p-capacité variationnelle, $p \in (1, \infty)$, d'un compact $K \subset \mathbb{R}^N$, $N \geq 2$, dénotée $c_p(K)$, est d'un usage fréquent en théorie des solutions des équations elliptiques non linéaires [1]. Elle possède en effet les deux propriétés suivantes :

1. dénotant k la dimension de K, éventuellement au sens de Hausdorff, on a la règle de nullité suivante :

$$c_p(K) = 0 \quad \text{si et seulement si } p \leq N - k;$$

2. sous des hypothèses appropriées, une solution d'une équation elliptique, définie dans un sous-domaine $\Omega \setminus K$, peut être prolongée au domaine Ω tout entier, en conservant la même régularité, dès lors que $c_p(K) = 0$. Autrement dit, K est une singularité éliminable dès lors que sa p-capacité variationnelle est nulle [79, 80].

Le point de vue adopté ici est inverse, puisque l'on cherche à détecter des compacts K en s'appuyant sur le fait que leur p-capacité est *strictement positive*, y compris pour des compacts de codimension ≥ 2, comme des points en 2D ou des courbes en 3D, dès lors que la valeur de p est suffisamment grande.

Se plaçant dans une perspective applicative, il est apparu pertinent de retenir ici une notion de capacité situant le compact K dans un domaine borné Ω. On étudiera donc ici la *p-capacité d'un condensateur* (K, Ω), telle que définie dans [49], et notée $C_p(K, \Omega)$ pour la distinguer de la notion de capacité variationnelle $c_p(K)$ qui considère le compact K dans $\mathbb{R}^N$.

L'objectif est donc d'obtenir des estimations de p-capacités de condensateurs strictement positives et lorsque c'est possible, d'en donner le développement asymptotique au sens du développement (5.6.1), notamment lorsque l'obstacle est un point ou une courbe. Une courbe régulière pouvant s'approximer localement par un segment, on se concentrera sur la cas où l'obstacle est un segment.

La plupart des estimations de capacités disponibles dans la littérature concerne le cas dit *harmonique* ou *électrostatique*, c'est à dire $p = 2$, qu'il s'agissent de résultats ayant trait à des capacités variationnelles comme en [85, 43] ou à des capacités de condensateurs comme dans [73, 74]. La plupart des estimations est obtenu sous forme d'inégalités. Les égalités sont l'exception. En ce qui concerne les p-capacités de condensateurs, $p \neq 2$, seules ont pu être calculées jusqu'ici les capacités de condensateurs sphériques (cf. [49] §2.11).

De plus on s'attend intuitivement à ce que la capacité d'un condensateur dépende non seulement de l'obstacle K, mais également de la géométrie du domaine Ω et de la position de l'obstacle dans le domaine. A l'inverse des capacités variationnelles, les cas de nullité des capacités de condensateurs ne sont que partiellement connus.

Concernant les problèmes de p-Laplace, la plupart des résultats disponibles s'intéressent à des singularités isolées comme dans [54, 90]. Des solutions anisotropes, de la forme $u(x) = |x|^\lambda \ \omega(x/|x|)$, où $\lambda \in \mathbb{R}$ et où ω est défini sur la sphère unité, ont été étudiées pour des équations quasilinéaires avec condition de Dirichlet, dans des domaines dont la frontière présente un point conique [86, 75]. Mais l'anisotropie induite dans l'équation de p-Laplace par un segment n'a pas encore été étudiée. Nous donnerons plusieurs illustrations de la forte anisotropie de la perturbation engendrée par un segment.

Dans ce chapitre, après une section préliminaire 7.2 dans laquelle nous étudions le cas d'obstacles d'intérieur non vide puis celui d'obstacles ponctuels, nos principales contributions, à la section 7.3, concernent le cas où l'obstacle est un segment. Nous introduirons pour ce faire la définition de *condensateurs équidistants* puis considérerons des *condensateurs elliptiques*.

7.2 Résultats préliminaires

7.2.1 Les p-capacités de condensateurs

Après Heinonen, Kilpeläinen et Martio [49], on pose

Définition 7.2.1. Soit $W(K,\Omega) := \{v \in C_0^\infty(\Omega) : v \geq 1 \text{ dans } K\}$. On définit

$$C_{p,N}(K,\Omega) := \inf_{v \in W(K,\Omega)} \int_\Omega |\nabla v|^p. \tag{7.2.1}$$

Le nombre positif ou nul $C_{p,N}(K,\Omega)$ est appelé la *p-capacité du condensateur* (K,Ω).

De façon équivalente ([49], p. 27), on peut remplacer l'ensemble $W(K,\Omega)$ dans la définition 7.2.1 par l'ensemble plus grand

$$W_0(K,\Omega) := \left\{v \in W_0^{1,p}(\Omega) \cap C(\Omega) : v \geq 1 \text{ dans } K\right\}.$$

Le compact K est appelé l'*obstacle* du condensateur. Une fonction $v \in W_0(K,\Omega)$ est dite *admissible* pour le condensateur.

Les p-capacités de condensateurs satisfont à l'axiomatique de Choquet [36]. En particulier :

Théorème 7.2.1. La fonction d'ensembles $K \to C_p(K,\Omega)$, où K est un compact inclus dans le domaine $\Omega \subset \mathbb{R}^N$, vérifie :

1. (Monotonie) Si $K_1 \subset K_2 \subset \Omega$ alors $C_p(K_1,\Omega) \leq C_p(K_2,\Omega)$.
2. (Monotonie) Si $K \subset \Omega_1 \subset \Omega_2$ alors $C_p(K,\Omega_2) \leq C_p(K,\Omega_1)$.
3. (Continuité descendante) Si $(K_n)_{n\geq 0}$ est une suite décroissante de compacts de Ω, i.e. $\Omega \supset K_0 \supset K_1 \supset \cdots \supset K_n \supset K_{n+1} \supset \cdots$ et si $K := \bigcap_{n\geq 0} K_n$, alors

$$C_p(K,\Omega) = \lim_{n\to+\infty} C_p(K_n,\Omega).$$

Ces trois propriétés seront d'un usage constant.

7.2.2 Estimation de la p-capacité par un problème de p-Laplace

On suppose dans cette sous-section 7.2.2 que Ω et K ont des frontières régulières de classe C^1. On considère le problème suivant de p-Laplace dans $\Omega \setminus K$ avec condition frontière de Dirichlet :

$$\begin{cases} -\Delta_p(u) = 0, & \text{dans } \Omega \setminus K \ , \\ u = 1, & \text{sur } \partial K \ , \\ u = 0, & \text{sur } \partial\Omega \ , \end{cases} \tag{7.2.2}$$

où Δ_p désigne l'opérateur p-Laplacien, $\Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

Il est bien connu [60, 17, 87, 91] que le problème (7.2.2) admet une unique solution $u \in W^{1,p}(\Omega \setminus K)$, qui est continue dans $\overline{\Omega \setminus K}$ (après redéfinition dans un ensemble de mesure nulle) et continûment différentiable dans $\Omega \setminus K$. En particulier on a ponctuellement $u = 0$ sur $\partial\Omega$ et $u = 1$ sur ∂K. De plus

$$0 < u(x) < 1, \quad \forall x \in \Omega \setminus K.$$

Proposition 7.2.2. Avec les notations précédentes, et avec l'hypothèse de régularité C^1 des frontières de Ω et de K, on a

$$C_p(K,\Omega) = \int_{\Omega\setminus K} |\nabla u|^p. \tag{7.2.3}$$

7.2.3 Cas d'un obstacle d'intérieur non vide

Soit un point $x_0 \in \Omega$. Soit les deux rayons strictement positifs

$$R_1 := \sup\{R > 0; B(x_0,R) \subset \Omega\} > 0 \quad \text{et} \quad R_2 := \inf\{R > 0; \Omega \subset B(x_0,R)\}.$$

Soit un domaine d'intérieur non vide $\omega \subset \mathbb{R}^N$ tel que $0 \in \omega$. Soit les deux nombres strictement positifs

$$\rho_1 := \sup\{\rho > 0; B(x_0,\rho) \subset \omega\} \quad \text{et} \quad \rho_2 := \inf\{\rho > 0; \omega \subset B(x_0,\rho)\}.$$

Enfin soit $\omega_\varepsilon := x_0 + \varepsilon \cdot \omega \subset B(x_0,R_1)$ pour $\varepsilon > 0$ assez petit et soit le condensateur $(\overline{\omega}_\varepsilon, \Omega)$ comme sur la Figure 7.1.

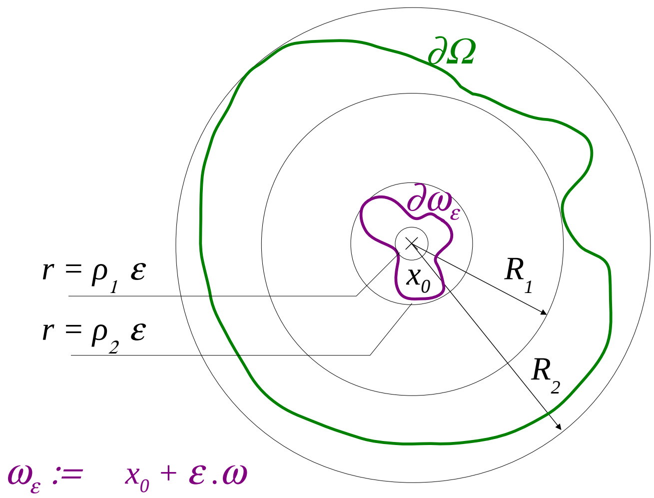

FIGURE 7.1 – Un condensateur dont l'obstacle $\overline{\omega}_\varepsilon$ a un intérieur non vide et un diamètre tendant vers zéro.

Proposition 7.2.3. Quand $\varepsilon \to 0$, on a les encadrements asymptotiques suivants :

1. Si $p = N$, alors :

$$-A^{N-1}(p-1)\log(R_2/\rho_1)\,[-\log\varepsilon]^{-p} + o\left([\log\varepsilon]^{-p}\right)$$

$$\leq \quad C_p(\overline{\omega}_\varepsilon,\Omega) - A^{N-1}\,[-\log\varepsilon]^{1-p} \quad \leq$$

$$-A^{N-1}(p-1)\log(R_1/\rho_2)\,[-\log\varepsilon]^{-p} + o\left([\log\varepsilon]^{-p}\right).$$

2. Si $p > N$, alors :

$$A^{N-1}\beta^{p-1}R_2^{N-p}\left[1+(p-1)\left(\rho_1\,\varepsilon/R_2\right)^{\beta}+o\left(\varepsilon^{\beta}\right)\right]$$

$$\leq \quad C_p(\overline{\omega}_\varepsilon,\Omega) \quad \leq$$

$$A^{N-1}\beta^{p-1}R_1^{N-p}\left[1+(p-1)\left(\rho_2\,\varepsilon/R_1\right)^{\beta}+o\left(\varepsilon^{\beta}\right)\right].$$

3. Si $p < N$, alors :

$$A^{N-1}\left(-\beta\right)^{p-1}\left(\rho_1\,\varepsilon\right)^{N-p}\left[1+(p-1)\left(\rho_1\,\varepsilon/R_2\right)^{-\beta}+o\left(\varepsilon^{-\beta}\right)\right]$$

$$\leq \quad C_p(\overline{\omega}_\varepsilon,\Omega) \quad \leq$$

$$A^{N-1}\left(-\beta\right)^{p-1}\left(\rho_2\,\varepsilon\right)^{N-p}\left[1+(p-1)\left(\rho_2\,\varepsilon/R_1\right)^{-\beta}+o\left(\varepsilon^{-\beta}\right)\right].$$

A noter qu'aucune hypothèse n'est nécessaire dans la Proposition 7.2.3 quant à la régularité des frontières de ω et de Ω.

Remarque 7.2.4. Les encadrements asymptotiques de la Proposition 7.2.3 fournissent des développements asymptotiques topologique au sens du développement (5.6.1).

1. Si $p = N$, alors on a

$$C_p(\overline{\omega}_\varepsilon,\Omega) = A^{N-1}\left[-\log\varepsilon\right]^{1-p}+o\left(\left[-\log\varepsilon\right]^{1-p}\right).$$

   Le gradient topologique est égal à A^{N-1}. Il est constant dans Ω. Il ne dépend ni de la forme du compact $\overline{\omega}$ ni de celle du domaine Ω.

2. Si $p < N$ et si ω est la boule unité, alors

$$C_p(\overline{B}_\varepsilon,\Omega) = A^{N-1}\left(-\beta\right)^{p-1}\varepsilon^{N-p}+o\left(\varepsilon^{N-p}\right).$$

   Le gradient topologique est égal à $A^{N-1}\left(-\beta\right)^{p-1}$. Il est constant dans Ω. Il ne dépend pas de la forme du domaine Ω.

3. Dans le cas harmonique $p = 2$ et pour les dimensions $N = 2$ ou $N = 3$, les résultats ci-dessus sont conformes à ceux obtenus antérieurement dans [47] pour l'équation de Laplace avec condition frontière de Dirichlet.

Remarque 7.2.5. D'après les encadrements établis à la Proposition 7.2.3, le domaine Ω, au travers des paramètres R_1 et R_2, n'influe pas sur le premier terme du développement asymptotique de la capacité $C_p(\overline{\omega}_\varepsilon,\Omega)$ lorsque $p \leq N$.

En revanche quand $p > N$, la position de x_0 dans Ω et la forme de Ω déterminent le premier ordre du développement au travers des paramètres R_1 et R_2. Cette situation illustre une différence importante entre le concept de capacité de condensateur (obstacle placé dans un domaine borné) et celui de capacité variationnelle (obstacle placé dans $\mathbb{R}^N$).

7.2.4 Capacité d'un point, estimations et vitesse de convergence

Proposition 7.2.6. Soit x_0 un point du domaine borné $\Omega \subset \mathbb{R}^N$. On a la règle de positivité suivante :

$$C_p(\{x_0\}, \Omega) > 0 \quad \text{si et seulement si} \quad p > N. \tag{7.2.4}$$

De plus, si $p > N$, alors :

$$A^{N-1}\beta^{p-1}R_2^{N-p} \leq C_p(\{x_0\}, \Omega) \leq A^{N-1}\beta^{p-1}R_1^{N-p}, \tag{7.2.5}$$

où $R_1 := \sup\{R > 0; B(x_0, R) \subset \Omega\}$ et $R_2 := \inf\{R > 0; \Omega \subset B(x_0, R)\}$.
En particulier, si $p > N$ et si $\Omega = B(x_0, R)$, alors on a l'égalité

$$C_p(\{x_0\}, B_R) = A^{N-1}\beta^{p-1}R^{N-p}.$$

Il découle de la propriété de continuité descendante du Théorème 7.2.1 que la capacité du point peut être approximée par la capacité d'une boule dont le rayon tend vers zéro. Sous cet angle, la vitesse de la convergence descendante devient un sujet d'intérêt quand $p > N$.

Proposition 7.2.7. Si $p > N$, pour tout $r \in (0, R)$, on a

$$C_p(\overline{B}(x_0, r), B(x_0, R)) - C_p(\{x_0\}, B(x_0, R)) = O\left(r^\beta\right). \tag{7.2.6}$$

Pour $p > N \geq 2$, on a $0 < \beta < 1$. Ainsi la vitesse de convergence en $O\left(r^\beta\right)$ est lente quand $r \to 0$. On montre même que

$$\lim_{\varepsilon \to 0} \frac{dC_p(\varepsilon, R)}{d\varepsilon} = +\infty.$$

7.3 Capacité d'un segment

Dans cette section, un segment sera un compact $S_\varepsilon \subset \mathbb{R}^N$, $N \geq 2$, défini par

$$S_\varepsilon := \{x_0 + s\tau \,;\, s \in [-\varepsilon/2, \varepsilon/2]\}$$

où x_0 est le centre du segment, $\varepsilon > 0$ sa longueur et $\tau \in \mathbb{R}^N$ un vecteur unitaire.

7.3.1 Condensateurs équidistants

La notion de condensateurs équidistants est introduite pour permettre certaines estimations de la p-capacité d'un segment. Pour tout $x \in \mathbb{R}^N$ et tout sous-ensemble $E \subset \mathbb{R}^N$, on note $d(x, E) = \inf\{|y - x| \,;\, y \in E\}$. D'après la Figure 7.2, on pose la

Définition 7.3.1. Soit $0 < \eta < R$. Soit le compact $K_\eta := \left\{x \in \mathbb{R}^N \mid d(x, S_\varepsilon) \leq \eta\right\}$ et le domaine borné $\Omega_R := \left\{x \in \mathbb{R}^N \mid d(x, S_\varepsilon) < R\right\}$. On dit que (K_η, Ω_R) est un *condensateur équidistant* dérivé du segment S_ε.

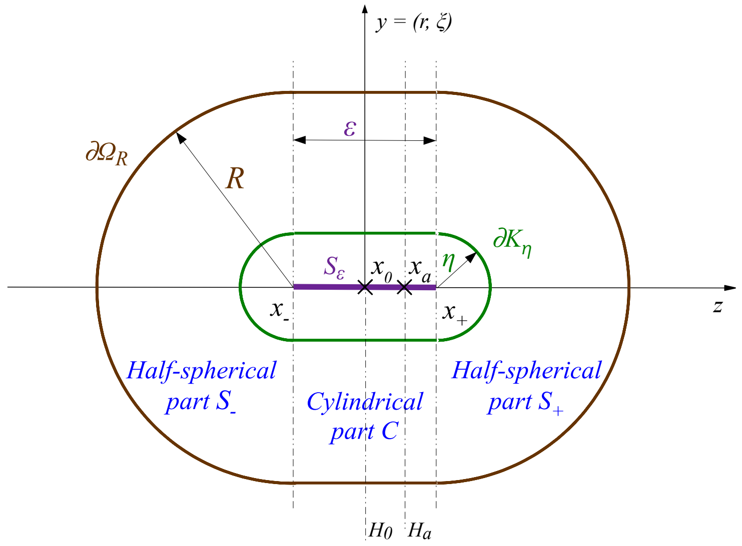

FIGURE 7.2 – Un condensateur équidistant (K_η, Ω_R).

Les condensateurs équidistants sont introduits ici en raison de leur propriétés géométriques remarquables. Les trois principales sont les suivantes :

1. La propriété de continuité descendante du Théorème 7.2.1 s'écrit
$$\lim_{\eta \to 0} C_p\left(K_\eta, \Omega_R\right) = C_p\left(S_\varepsilon, \Omega_R\right).$$
2. Soit v une fonction admissible pour le condensateur (K_η, Ω_R). Alors l'énergie de v dans les deux demi-hémisphères S_- et S_+ est minorée par une capacité de condensateur sphérique en dimension N, c'est à dire :
$$C_p(B_\eta, B_R) \leq \int_{S_- \cup S_+} |\nabla v|^p .$$
3. Soit H_a un hyperplan orthogonal au segment S_ε et intersectant celui-ci. Alors
$$(K_\eta \cap H_a, \Omega_R \cap H_a)$$
est un condensateur sphérique en dimension $N-1$.

Nous donnons tout d'abord deux illustrations de la forte anisotropie induite par le segment dans le problème de p-Laplace.

1. La définition 7.2.1 de la p-capacité d'un condensateur permet d'obtenir facilement des majorants en calculant l'énergie de fonctions admissibles. L'obtention d'un minorant ne peut suivre un cheminement aussi direct. Dans [73, 74], G. Pólya et G. Szegö ont montré que dans le cas harmonique $p = 2$, la symétrisation de Schwarz permet d'obtenir un minorant à une 2-capacité de condensateur. Plus récemment, Brothers et Ziemer [32, 42, 33] ont étendu ce résultat, connu sous le nom d'inégalité de réarrangement de Pólya-Szegö pour les intégrales de type Dirichlet.

Appliquant cette inégalité de Pólya-Szegö à un condensateur équidistant (K_η, Ω_R), puis appliquant la propriété de continuité descendante quand $\eta \to 0$, on obtient la minoration suivante

$$C_p(\{0\}, B_{R^\sharp}) \leq C_p(S_\varepsilon, \Omega_R). \tag{7.3.1}$$

où

$$R^\sharp := \left[R^{N-1} \left(R + \frac{A^{N-2}}{A^{N-1}} \varepsilon \right) \right]^{\frac{1}{N}} > R.$$

Cette minoration n'apporte en réalité aucune information complémentaire :

(a) quand $p \leq N$ le minorant $C_p(\{0\}, B_{R^\sharp})$ est nul, d'après la Proposition 7.2.6 ;

(b) quand $p > N$, on savait déjà par monotonie que

$$C_p(\{0\}, B_{R^\sharp}) \leq C_p(\{0\}, B_R) \leq C_p(S_\varepsilon, B(x_0, R)) \leq C_p(S_\varepsilon, \Omega_R).$$

L'inégalité de réarrangement de Pólya-Szegö pour les intégrales de type Dirichlet ne permet donc pas de cerner l'effet anisotrope causé par le segment dans l'équation de p-Laplace.

2. La propriété de continuité descendante indique que

$$\lim_{\varepsilon \to 0} C_p\left(S_\varepsilon, B(x_0, R)\right) = C_p\left(\{x_0\}, B(x_0, R)\right).$$

En supposant que la longueur ε du segment S_ε est suffisamment petite, une idée pourrait être, dans le cas $p > N$, d'essayer de construire une fonction admissible pour le segment à partir de la fonction $s_{p,N}$ minimisant l'énergie du condensateur $(\{x_0\}, B(x_0, R))$.
On considère ainsi la fonction $\bar{u} : \overline{\Omega_R \setminus S_\varepsilon} \to \mathbb{R}$ définie par :

$$\begin{cases} \text{si } x \in \overline{S}_- \cap \{z < -\varepsilon/2\} & \text{alors } \bar{u}(x) := s_{p,N}(\rho_-) \text{ avec } \rho_- = |x - A| \ , \\ \text{si } x \in \overline{S}_+ \cap \{z > \varepsilon/2\} & \text{alors } \bar{u}(x) := s_{p,N}(\rho_+) \text{ avec } \rho_+ = |x - B| \ , \\ \text{si } x \in \overline{C} & \text{alors } \bar{u}(x) := s_{p,N}(r) \text{ avec } r = |y|, \end{cases}$$

comme indiqué sur la Figure 7.3.
On pourrait s'attendre à ce que l'énergie de $\bar{u}$ fournisse une bonne approximation de $C_p(S_\varepsilon, \Omega_R)$, pour ε petit, puisque par construction l'énergie localisée dans les deux parties hémisphériques vaut

$$\int_{S_- \cup S_+} |\nabla \bar{u}(x)|^p \ dx \quad = \quad C_p(\{x_0\}, B(x_0, R)).$$

En réalité $\bar{u}$ n'est même pas admissible pour le condensateur $(S_\varepsilon, \Omega_R)$. Calculant l'énergie de $\bar{u}$ dans la partie cylindrique du condensateur, il vient

$$\int_C |\nabla \bar{u}(x)|^p \ dx = \int_C |\nabla s_{p,N}(x)|^p \ dx = \varepsilon \ A^{N-2} \frac{\beta^p}{R^{\beta p}} \int_0^R r^{(\beta-1)p} r^{N-2} \ dr = +\infty,$$

l'exposant $(\beta - 1)p + N - 2 \in (-2, -1)$ faisant que l'intégrale diverge en 0.

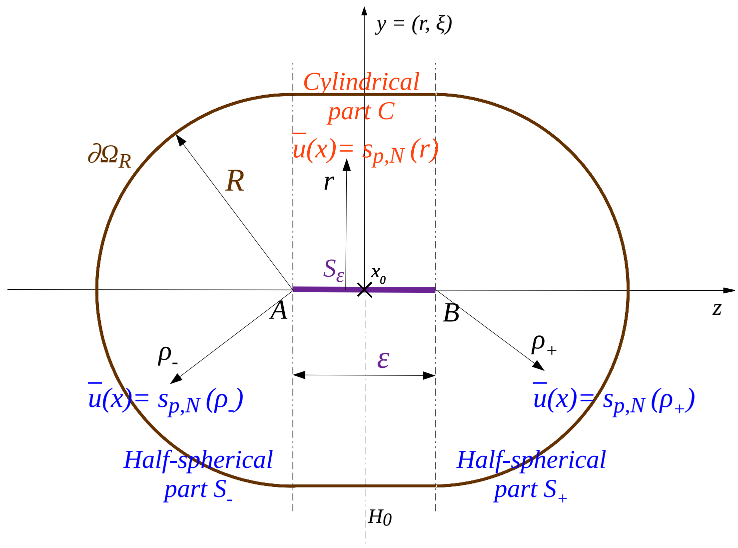

FIGURE 7.3 – Un faux vrai candidat $\bar{u}$ pour le condensateur $(S_\varepsilon, \Omega_R)$.

On peut conclure qualitativement qu'en dépit de la propriété de continuité descendante, la solution minimisant l'énergie du condensateur, si elle existe, subit une brutale réorganisation spatiale quand ε passe de 0 à une valeur strictement positive arbitrairement petite. Ce qui intervient ici est moins la longueur ε du segment que la discontinuité de la dimension de l'obstacle, qui passe de 0 à 1. Cet exemple illustre les relations étroites existant entre mesures de Hausdorff et capacités.

Quand $p > N$, on ne peut donc simplement construire une solution admissible pour le segment par extension de la solution minimisant l'énergie d'un condensateur dont l'obstacle est réduit à un point.

Les propriétés géométriques des condensateurs équidistants permettent d'obtenir le résultat suivant qui met en exergue le lien existant entre la capacité d'un segment en dimension N d'une part et la capacité d'un point en dimension N et la capacité d'un point en dimension $N-1$ d'autre part.

Théorème 7.3.1. Soit Ω un domaine borné de $\mathbb{R}^N$ et $x_0 \in \Omega$. Soit

$$R := \sup \{|y - x_0| \, ; y \in \Omega\} \in (0, +\infty).$$

Soit S_ε un segment centré sur le point x_0 et de longueur $\varepsilon > 0$ telle que $S_\varepsilon \subset \Omega$. Alors on a la minoration

$$C_{p,N}(S_\varepsilon, \Omega) \;\geq\; C_{p,N}(\{x_0\}, B_R) + \varepsilon\, C_{p,N-1}(\{x_0\}, B_R). \tag{7.3.2}$$

où $C_{p,N}(\{x_0\}, B_R)$ (resp. $C_{p,N-1}(\{x_0\}, B_R)$ désigne la p-capacité du point $\{x_0\}$ dans la boule $B(x_0, R) \subset \mathbb{R}^N$ (resp. la p-capacité du point $\{x_0\}$ dans la boule $B_{N-1}(x_0, R) \subset \mathbb{R}^{N-1}$).

D'après la Proposition 7.2.6, trois cas doivent être distingués concernant le minorant de l'inégalité (7.3.2)

– Si $N-1 < p \leq N$, on a $C_{p,N}(\{x_0\}, B_R) = 0$ et $C_{p,N-1}(\{x_0\}, B_R) > 0$. L'inégalité s'écrit :

$$\varepsilon\, C_{p,N-1}(\{x_0\}, B_R) \leq C_{p,N}(S_\varepsilon, \Omega).$$

En particulier on a $C_{p,N}(S_\varepsilon, \Omega) > 0$.

– Si $p > N$, les deux capacités $C_{p,N}(\{x_0\}, B_R)$ et $C_{p,N-1}(\{x_0\}, B_R)$ sont strictement positives. A nouveau on a $C_{p,N}(S_\varepsilon, \Omega) > 0$.

– Si $p \leq N-1$, les deux capacités $C_{p,N}(\{x_0\}, B_R)$ et $C_{p,N-1}(\{x_0\}, B_R)$ sont nulles.

Ainsi la condition $p > N-1$ implique que la stricte positivité $C_{p,N}(S_\varepsilon, \Omega) > 0$. La réciproque s'obtient en recourant à nouveau aux condensateurs équidistants. On a donc le

Théorème 7.3.2. La p-capacité de condensateur d'un segment S_ε de longueur $\varepsilon > 0$ inclus dans un domaine borné $\Omega \subset \mathbb{R}^N$ est strictement positive si et seulement si $p > N-1$.

7.3.2 Condensateurs elliptiques

Nous poursuivons les estimations de p-capacités de condensateur d'un segment en introduisant les *condensateurs elliptiques* (cf. Figure 7.4).

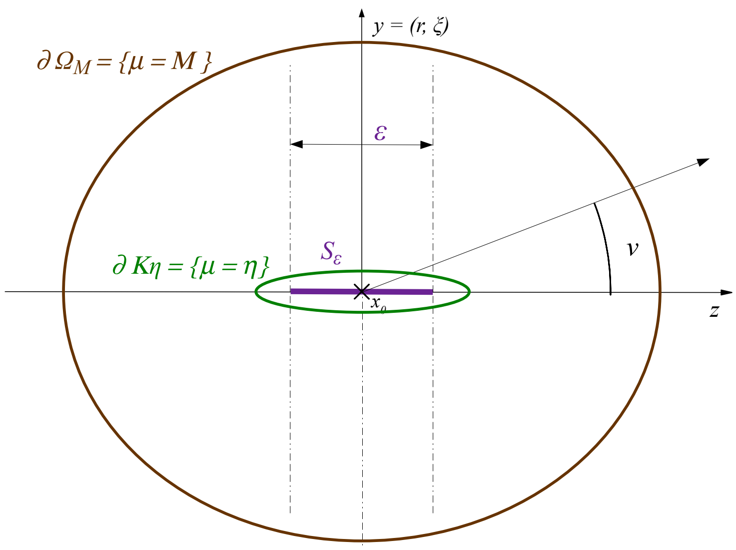

FIGURE 7.4 – Un condensateur elliptique (K_η, Ω_M).

On considère d'abord des coordonnées cylindriques $(z, y) = (z, r, \xi)$ dans $\mathbb{R}^N$, avec $z \in \mathbb{R}$, $y = r\xi \in \mathbb{R}^{N-1}$, $r \geq 0$ et $\xi \in S^{N-2}$. Puis on considère des coordonnées elliptiques (μ, ν, ξ) [55, 92, 93] définies implicitement pour $\mu \in [0, +\infty)$, $\nu \in [0, \pi]$ et $\xi \in S^{N-2}$ par

$$\begin{cases} z(\mu,\nu) & := \varepsilon/2\ \cosh\mu\cos\nu, \\ r(\mu,\nu) & := \varepsilon/2\ \sinh\mu\sin\nu, \\ \xi & := \xi. \end{cases} \tag{7.3.3}$$

Le segment S_ε est défini en cordonnées elliptiques par $S_\varepsilon = \{\mu = 0, \nu \in [0,\pi]\}$. Il est vu ici comme la limite d'un ellipsoïde dont l'excentricité tend vers 1 quand $\mu \to 0$.

Définition 7.3.2. Soit $0 < \eta < M$. Soit le domaine borné

$$\Omega_M := \left\{x = (\mu,\nu,\xi) \in \mathbb{R}^N \ ;\ 0 \le \mu < M, \nu \in [0,\pi]\,, \xi \in S^{N-2}\right\}$$

et le compact

$$K_\eta := \left\{x = (\mu,\nu,\xi) \in \mathbb{R}^N \ ;\ 0 \le \mu \le \eta, \nu \in [0,\pi]\,, \xi \in S^{N-2}\right\}.$$

On dit que (K_η, Ω_M) est un *condensateur elliptique* dérivé du segment S_ε.

Dans le cas harmonique $p = 2$, le problème (7.2.2) appliqué au condensateur (K_η, Ω_M) est à variables séparables en coordonnées elliptiques. Le calcul explicite de la solution permet d'obtenir une estimation de la 2-capacité de condensateur du segment dans le plan (pour mémoire, en dimension supérieure, cette 2-capacité est nulle).

Proposition 7.3.3. Soit $0 < \varepsilon/2 < R$. Soit S_ε un segment centré en un point x_0, de longueur ε et inclus dans $B_R = B(x_0, R) \subset \mathbb{R}^2$. Alors on a l'encadrement suivant :

$$\frac{2\pi}{\log\left(2R/\varepsilon + \sqrt{1 + 4R^2/\varepsilon^2}\right)} \le C_2(S_\varepsilon, B_R) \le \frac{2\pi}{\log\left(2R/\varepsilon + \sqrt{-1 + 4R^2/\varepsilon^2}\right)}. \tag{7.3.4}$$

Lorsque ε tend vers 0, il vient le développement asymptotique suivant.

Corollaire 7.3.4. Dans les hypothèses de la Proposition 7.3.3, pour $\varepsilon > 0$ assez petit, on a

$$C_2(S_\varepsilon, \Omega) = \frac{-2\pi}{\log \varepsilon} + o\left(\frac{-1}{\log \varepsilon}\right). \tag{7.3.5}$$

Remarque 7.3.5.
- Le développement (7.3.5) est un développement asymptotique topologique au sens de (5.6.1), pour l'équation de Laplace avec condition frontière de Dirichlet en 2D. La perturbation de l'équation de Laplace en 2D avec condition de Neumann homogène sur un segment, a été étudiée dans [9].
- Depuis [47], il est connu que dans le cas de l'équation de Laplace avec condition frontière de Dirichlet en 2D, le premier ordre du développement asymptotique topologique ne dépend pas de la forme de l'obstacle, pour des *obstacles d'intérieurs non vides.*

  Or comparant le développement 7.3.5 ci-dessus avec le développement asymptotique fourni par la Proposition 7.2.3 pour un condensateur sphérique $(B_{\frac{\varepsilon}{2}}, B_R)$ et appliquant la propriété de monotonie du Théorème 7.2.1, on obtient que

  $$C_2(K_\varepsilon, B_R) = \frac{-2\pi}{\log \varepsilon} + o\left(\frac{-1}{\log \varepsilon}\right), \text{ pour tout compact } K_\varepsilon,\ S_\varepsilon \subset K_\varepsilon \subset B_{\frac{\varepsilon}{2}}. \tag{7.3.6}$$

  Ainsi le développement asymptotique topologique ne dépend pas au premier ordre de la forme de K_ε, même quand K_ε est réduit au segment S_ε. Il ne dépend que de la taille ε de l'obstacle. Il s'ensuit que le gradient topologique de la 2-capacité

de condensateur en 2D n'est pas un outil approprié pour distinguer les courbes des obstacles d'intérieurs non vides.
Comment pourra-t-on surmonter cet inconvénient ? peut-être en choisissant des valeurs différentes de p ? peut-être en effectuant le tri des obstacles sur la base du terme de deuxième ordre du développement asymptotique, ce qui voudrait dire considérer la *hessienne topologique* ?
Bien qu'on ne connaisse pas encore le développement asymptotique de $C_p(S_\varepsilon, B_R)$ en dimension supérieure $N \geq 3$, une difficulté similaire est susceptible d'apparaître dans le cas $p = N \geq 3$. En effet, il découle de la Proposition 7.2.3 que, lorsque $p = N$, le développement asymptotique topologique ne dépend jamais de la forme d'un obstacle d'intérieur non vide.

Dans le cas général $p \neq 2$, les condensateurs elliptiques semblent constituer un cadre favorable à l'obtention d'autres estimations de la p-capacité de condensateur d'un segment. La coordonnée angulaire ν permet pour ainsi dire de faire varier continûment la dimension dans laquelle opère l'équation de p-Laplace, de N pour $\nu = 0$ à $N-1$ pour $\nu = \pi/2$ puis de nouveau N pour $\nu = \pi$. La géométrie du problème se trouve simplifiée du fait que solutions et intégrales sont à calculer dans le rectangle $\mathcal{R} := [\eta, M] \times [0, \pi]$.

7.4 Conclusions

Dans ce chapitre 7, nous avons étudié des estimations et lorsque cela est possible, des développements asymptotiques de p-capacités de condensateur. En liaison avec de potentielles applications, nous avons en particulier approfondi les cas dans lesquels l'obstacle du condensateur est un point ou un segment, et leurs approximations par des capacités de condensateurs dont les obstacles sont d'intérieurs non vides.

Dans la section préliminaire 7.2, nous avons montré que lorsque les bords du condensateur sont réguliers, la p-capacité peut être obtenue en calculant l'énergie de la solution d'un problème de p-Laplace avec condition frontière de Dirichlet. Puis nous avons donné un encadrement asymptotique de la p-capacité d'un condensateur, quand son obstacle d'intérieur non vide voit son diamètre tendre vers 0.

A la section 7.2.4, nous avons établi directement que la capacité d'un obstacle ponctuel est strictement positive si et seulement si $p > N$. Dans ce dernier cas, il est naturel de chercher à approximer cette capacité par celle d'un condensateur sphérique, dont l'obstacle a un rayon suffisamment petit. Il s'avère que la vitesse de convergence descendante de la capacité d'une boule vers celle du point est lente, en r^β, avec $0 < \beta < 1$.

En venant à la question de la p-capacité de condensateur d'un segment à la section 7.3, nous avons introduit les *condensateurs équidistants.* Nous avons illustré de deux manières la forte anisotropie introduite par un obstacle en forme de segment dans l'équation de p-Laplace. En premier lieu, l'inégalité de réarrangement de Pólya-Szegö appliquée aux intégrales de type Dirichlet aboutit à une minoration triviale. En second lieu et pour $p > N$, aussi petite soit la longueur d'un segment, il n'est pas possible de construire une fonction admissible par la simple extension de la solution minimisant l'énergie d'un condensateur dont l'obstacle est réduit à un point.

Grâce aux condensateurs équidistants, nous avons mis en évidence une minoration de la capacité du segment qui fait intervenir la p-capacité de condensateur d'un point en dimension N et la p-capacité de condensateur d'un point en dimension $N-1$. Puis nous avons prouvé par cette méthode la règle de stricte positivité de la p-capacité de condensateur d'un segment. La méthode des condensateurs équidistants pourrait être étendue à des obstacles de codimension ≥ 2 en dimensions supérieures.

Introduisant alors les *condensateurs elliptiques*, nous avons établi une estimation et le développement asymptotique topologique de la 2-capacité d'un segment dans le plan. Par comparaison avec les résultats obtenus dans le cas d'obstacles d'intérieurs non vide, il apparaît que le premier terme du développement asymptotique ne dépend pas de la forme de l'obstacle mais seulement de sa taille, même si celui-ci est réduit à un segment. Ainsi le gradient topologique de la 2-capacité de condensateur en 2D ne permet pas de distinguer les courbes d'une part et les obstacles d'intérieurs non vides d'autre part. Il faudra surmonter cette difficulté, soit en considérant d'autres valeurs du paramètre p, soit en s'intéressant au second ordre du développement asymptotique topologique, c'est à dire à la *hessienne topologique*.

Nous avons enfin brièvement souligné les atouts dont disposent les condensateurs elliptiques pour permettre l'obtention d'autres estimations de la p-capacité de condensateur d'un segment.

A l'aune de ce chapitre, l'estimation de p-capacités de condensateurs pour des obstacles d'intérieur vide apparaît être un domaine difficile mais stimulant de recherche. Plusieurs questions importantes restent à approfondir :

- la vitesse de convergence de la continuité descendante ;
- la localisation de l'énergie dans le condensateur ;
- la question de l'application de méthodes de transport de mesures, plus récentes que l'inégalité de Pólya-Szegö, pour parvenir à saisir l'effet anisotrope induit par un segment dans un condensateur ;
- la possibilité d'utiliser un développement asymptotique topologique, au premier ou au second ordre, pour distinguer les courbes des obstacles d'intérieur non vide dans le plan ;
- et toutes les questions similaires pour des obstacles de codimension ≥ 2 en dimension supérieure $N \geq 3$.

En définitive, dans le sillage de la définition donnée par Hausdorff des dimensions non entières, la question de la p-capacité de condensateur d'un segment offre un intéressant cas pratique d'étude d'un opérateur différentiel, le p-laplacien, opérant entre deux dimensions, elles même reliées à deux directions orthogonales.

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Résumé : La Partie I présente l'obtention du développement asymptotique topologique pour une classe d'équations elliptiques quasilinéaires. Un point central réside dans la possibilité de définir la variation de l'état direct à l'échelle 1 dans $\mathbb{R}^N$. Après avoir défini un cadre fonctionnel approprié faisant intervenir les normes L^p et L^2, et avoir justifié la classe d'équations considérée, la méthode se poursuit par l'étude du comportement asymptotique de la solution du problème d'interface non linéaire dans $\mathbb{R}^N$ et par une mise en dualité appropriée des états direct et adjoint aux différentes étapes d'approximation.

La Partie II traite d'estimations et de développements asymptotiques de p-capacités de condensateurs, dont l'obstacle est d'intérieur vide et de codimension ≥ 2. Après les résultats préliminaires, les condensateurs équidistants permettent de donner deux illustrations de l'anisotropie engendrée par un segment dans l'équation de p-Laplace, puis d'établir une minoration de la p-capacité N-dimensionnelle d'un segment, qui fait intervenir les p-capacités d'un point, respectivement en dimensions N et $(N-1)$. Les condensateurs elliptiques permettent d'établir que le gradient topologique de la 2-capacité n'est pas un outil approprié pour distinguer les courbes des obstacles d'intérieur non vide en 2D.

Abstract : In Part I we provide topological asymptotic expansions for a class of quasilinear elliptic equations. A key point lies in the ability to define the variation of the direct state at scale 1 in $\mathbb{R}^N$. After setting up an appropriate functional framework involving both the L^p and the L^2 norms, and then justifying the chosen class of equations, the approach goes on with the study of the asymptotic behavior of the solution of the nonlinear interface problem in $\mathbb{R}^N$ and by setting up an adequate duality scheme between the direct and adjoint states at each step of approximation. Part II deals with estimates and asymptotic expansions of condenser p-capacities and focuses on obstacles with empty interiors and with codimensions ≥ 2. After preliminary results, equidistant condensers are introduced to point out the anisotropy caused by a segment in the p-Laplace equation, and to provide a lower bound to the N-dimensional condenser p-capacity of a segment, by means of the N-dimensional and of the (N-1)-dimensional condenser p-capacities of a point. Introducing elliptical condensers, it turns out that the topological gradient of the 2-capacity is not an appropriate tool to separate curves and obstacles with nonempty interior in 2D.